

The Ramsey number of mixed-parity cycles II

David G. Ferguson

Abstract

Denote by $R(G_1, G_2, G_3)$ the minimum integer N such that any three-colouring of the edges of the complete graph on N vertices contains a monochromatic copy of a graph G_i coloured with colour i for some $i \in 1, 2, 3$. In a series of three papers of which this is the second, we consider the case where G_1, G_2 and G_3 are cycles of mixed parity. Here and in the previous paper, we consider $R(C_n, C_m, C_\ell)$, where n and m are even and ℓ is odd. Figaj and Łuczak determined an asymptotic result for this case, which we improve upon to give an exact result. We prove that for n, m and ℓ sufficiently large

$$R(C_n, C_m, C_\ell) = \max\{2n + m - 3, n + 2m - 3, \tfrac{1}{2}n + \tfrac{1}{2}m + \ell - 2\}.$$

The proof of this result is mostly contained within the first paper in this series, however, in the case that the longest cycle is of odd length, we require an additional technical result, the proof of which makes up the majority of this paper.

For graphs G_1, G_2, G_3 , the Ramsey number $R(G_1, G_2, G_3)$ is the smallest integer N such that every edge-colouring of the complete graph on N vertices with up to three colours, results in the graph having, as a subgraph, a copy of G_i coloured with colour i for some i . We consider the case when G_1, G_2 and G_3 are cycles, specifically cycles of mixed parity. For a history of this problem see [Fer15a].

Defining $\langle\langle x \rangle\rangle$ to be the largest even number not greater than x and $\langle x \rangle$ to be the largest odd number not greater than x , Figaj and Łuczak [FL07b] proved that, for all $\alpha_1, \alpha_2, \alpha_3 > 0$,

$$\begin{aligned} \text{(i)} \quad & R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_3\}n + o(n), \\ \text{(ii)} \quad & R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{4\alpha_1, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_3\}n + o(n), \end{aligned}$$

as $n \rightarrow \infty$.

In [Fer15a], improving on their result, in the case when exactly one of the cycles is of odd length, we considered the following result:

Theorem A. *For every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2$, there exists a positive integer $n_A = n_A(\alpha_1, \alpha_2, \alpha_3)$ such that, for $n > n_A$,*

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{2\langle\langle \alpha_1 n \rangle\rangle + \langle\langle \alpha_2 n \rangle\rangle - 3, \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle + \langle \alpha_3 n \rangle - 2\}.$$

In order to prove Theorem A, we required a new Ramsey-stability result for so called connected-matchings. Owing to the length of the proof of this stability result, part of it was postponed to this paper. The stability result, referred to as Theorem B, will be restated in the next section, after we have given the necessary definitions.

The complementary case, that is, the case when exactly one of the cycles is of even length is considered in [Fer15c], where we prove the following result, which again improves upon the corresponding result of Figaj and Łuczak:

Theorem C. *For every $\alpha_1, \alpha_2, \alpha_3 > 0$, there exists a positive integer $n_C = n_C(\alpha_1, \alpha_2, \alpha_3)$ such that, for $n > n_C$,*

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{4\langle\langle \alpha_1 n \rangle\rangle, \langle\langle \alpha_1 n \rangle\rangle + 2\langle \alpha_2 n \rangle, \langle\langle \alpha_1 n \rangle\rangle + 2\langle \alpha_3 n \rangle\} - 3.$$

1 Connected-matching stability result

Before proceeding to restate Theorem B, we remind the reader of a few concepts and structures from [Fer15a].

We define a *matching* to be collection of pairwise vertex-disjoint edges. We will sometimes abuse terminology and, where appropriate, refer to a matching by its vertex set rather than its edge set. Recall also that we call a matching with all its vertices in the same component of G a *connected-matching* and that a connected-matching is called *odd* if the component containing the matching also contains an odd cycle. Note that we call a connected-matching with all its edges contained in a monochromatic component of G a *monochromatic connected-matching*.

We say that a graph $G = (V, E)$ on N vertices is a -almost-complete for $0 \leq a \leq N - 1$ if its minimum degree $\delta(G)$ is at least $(N - 1) - a$. Observe that, if G is a -almost-complete and $X \subseteq V$, then $G[X]$ is also a -almost-complete. We say that a graph G on N vertices is $(1 - c)$ -complete for $0 \leq c \leq 1$ if it is $c(N - 1)$ -almost-complete, that is, if $\delta(G) \geq (1 - c)(N - 1)$. Observe that, for $c \leq \frac{1}{2}$, any $(1 - c)$ -complete graph is connected.

We say that a bipartite graph $G = G[U, W]$ is a -almost-complete if every $u \in U$ has degree at least $|W| - a$ and every $w \in W$ has degree at least $|U| - a$. Notice that, if $G[U, W]$ is a -almost-complete and $U_1 \subseteq U, W_1 \subseteq W$, then $G[U_1, W_1]$ is a -almost-complete. We say that a bipartite graph $G = G[U, W]$ is $(1 - c)$ -complete if every $u \in U$ has degree at least $(1 - c)|W|$ and every $w \in W$ has degree at least $(1 - c)|U|$. Again, notice that, for $c < \frac{1}{2}$, any $(1 - c)$ -complete bipartite graph $G[U, W]$ is connected, provided that $U, W \neq \emptyset$.

We say that a graph G on N vertices is c -sparse for $0 < c < 1$ if its maximum degree is at most $c(N - 1)$. We say a bipartite graph $G = G[U, W]$ is c -sparse if every $u \in U$ has degree at most $c|W|$ and every vertex $w \in W$ has degree at most $c|U|$.

Given a three-coloured graph G , we refer to the first, second and third colours as red, blue and green respectively and use G_1, G_2, G_3 to refer to the monochromatic spanning subgraphs of G . That is, G_1 (resp. G_2, G_3) has the same vertex set as G and includes, as an edge, any edge which (in G) is coloured red (resp. blue, green). If G_1 contains the edge uv , we say that u and v are *red neighbours* of each other in G . Similarly, if $uv \in E(G_2)$, we say that u and v are *blue neighbours* and, if $uv \in E(G_3)$, we say that that u and v are *green neighbours*.

Definition 1.1. For x_1, x_2, c_1, c_2 positive, γ_1, γ_2 colours, let $\mathcal{H}(x_1, x_2, c_1, c_2, \gamma_1, \gamma_2)$ be the class of edge-multicoloured graphs defined as follows:

A given two-multicoloured graph $H = (V, E)$ belongs to \mathcal{H} if its vertex set can be partitioned into $X_1 \cup X_2$ such that

- (i) $|X_1| \geq x_1, |X_2| \geq x_2$;
- (ii) H is c_1 -almost-complete; and
- (iii) defining H_1 to be the spanning subgraph induced by the colour γ_1 and H_2 to be the subgraph induced by the colour γ_2 ,
 - (a) $H_1[X_1]$ is $(1 - c_2)$ -complete and $H_2[X_1]$ is c_2 -sparse,
 - (b) $H_2[X_1, X_2]$ is $(1 - c_2)$ -complete and $H_1[X_1, X_2]$ is c_2 -sparse.



Figure 1: $H \in \mathcal{H}(x_1, x_2, c_1, c_2, \text{red}, \text{blue})$.

Definition 1.2. For x_1, x_2, x_3, c positive, let $\mathcal{K}(x_1, x_2, x_3, c)$ be the class of edge-multicoloured graphs defined as follows:

A given three-multicoloured graph $H = (V, E)$ belongs to \mathcal{K} if its vertex set can be partitioned into $X_1 \cup X_2 \cup X_3$ such that

- (i) $|X_1| \geq x_1, |X_2| \geq x_2, |X_3| \geq x_3$;
- (ii) H is c -almost-complete; and
- (iii) all edges present in $H[X_1, X_3]$ are coloured exclusively red, all edges present in $H[X_2, X_3]$ are coloured exclusively blue, and all edges present in $H[X_3]$ are coloured exclusively green.

Definition 1.3. For x_1, x_2, y_1, y_2, z, c positive, let $\mathcal{K}^*(x_1, x_2, y_1, y_2, z, c)$ be the class of edge-multicoloured graphs defined as follows:

A given three-multicoloured graph $H = (V, E)$ belongs to \mathcal{K}^* , if its vertex set can be partitioned into $X_1 \cup X_2 \cup Y_1 \cup Y_2$ such that

- (i) $|X_1| \geq x_1, |X_2| \geq x_2, |Y_1| \geq y_1, |Y_2| \geq y_2, |Y_1| + |Y_2| \geq z$;
- (ii) H is c -almost-complete; and
- (iii) all edges present in $H[X_1, Y_1]$ and $H[X_2, Y_2]$ are coloured exclusively red, all edges present in $H[X_1, Y_2]$ and $H[X_2, Y_1]$ are coloured exclusively blue and all edges present in $H[X_1, X_2]$ and $H[Y_1, Y_2]$ are coloured exclusively green.

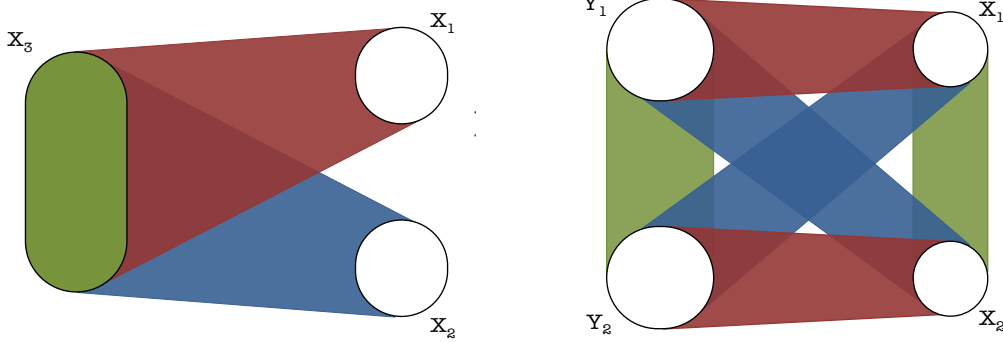


Figure 2: $H \in \mathcal{K}(x_1, x_2, x_3, c)$ and $H \in \mathcal{K}^*(x_1, x_2, y_1, y_2, c)$.

Having defined the coloured structures, we now restate Theorem B:

Theorem B. For every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2$, letting

$$c = \max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\},$$

there exists $\eta_B = \eta_B(\alpha_1, \alpha_2, \alpha_3)$ and $k_B = k_B(\alpha_1, \alpha_2, \alpha_3, \eta)$ such that, for every $k > k_B$ and every η such that $0 < \eta < \eta_B$, every three-multicolouring of G , a $(1 - \eta^4)$ -complete graph on

$$(c - \eta)k \leq K \leq (c - \frac{1}{2}\eta)k$$

vertices, results in the graph containing at least one of the following:

- (i) a red connected-matching on at least $\alpha_1 k$ vertices;
- (ii) a blue connected-matching on at least $\alpha_2 k$ vertices;
- (iii) a green odd connected-matching on at least $\alpha_3 k$ vertices;

(iv) disjoint subsets of vertices X and Y such that $G[X]$ contains a two-coloured spanning subgraph H from $\mathcal{H}_1 \cup \mathcal{H}_2$ and $G[Y]$ contains a two-coloured spanning subgraph K from $\mathcal{H}_1 \cup \mathcal{H}_2$, where

$$\begin{aligned}\mathcal{H}_1 &= \mathcal{H} \left((\alpha_1 - 2\eta^{1/32})k, (\tfrac{1}{2}\alpha_2 - 2\eta^{1/32})k, 3\eta^4k, \eta^{1/32}, \text{red}, \text{blue} \right), \\ \mathcal{H}_2 &= \mathcal{H} \left((\alpha_2 - 2\eta^{1/32})k, (\tfrac{1}{2}\alpha_1 - 2\eta^{1/32})k, 3\eta^4k, \eta^{1/32}, \text{blue}, \text{red} \right); \end{aligned}$$

(v) a subgraph from

$$\mathcal{K} \left((\tfrac{1}{2}\alpha_1 - 14000\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 14000\eta^{1/2})k, (\alpha_3 - 68000\eta^{1/2})k, 4\eta^4k \right);$$

(vi) a subgraph from $\mathcal{K}_1^* \cup \mathcal{K}_2^*$, where

$$\begin{aligned}\mathcal{K}_1^* &= \mathcal{K}^* \left((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k, \right. \\ &\quad \left. (\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4k \right), \\ \mathcal{K}_2^* &= \mathcal{K}^* \left((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\tfrac{3}{4}\alpha_3 - 140\eta^{1/2})k, \right. \\ &\quad \left. 100\eta^{1/2}k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4k \right). \end{aligned}$$

Furthermore,

(iv) occurs only if $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 14\eta^{1/2}$ with $H_1, H_2 \in \mathcal{H}_1$ unless $\alpha_2 \geq \alpha_1 - \eta^{1/16}$; and

(v) and (vi) occur only if $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}$.

This result forms a partially strengthened analogue of the main technical result of the paper of Figaj and Łuczak [FL07b]. In that paper, Figaj and Łuczak considered a similar graph but on slightly more than $\max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\}k$ vertices and proved the existence of a connected-matching, whereas we consider a graph on slightly fewer vertices and prove the existence of either a monochromatic connected-matching or a particular structure.

In [Fer15a, Section 7], we proved Theorem B, in the case that $\alpha_1 \geq \alpha_3$, the remainder of this paper is dedicated to proving it in the complementary case.

2 Definitions and tools

In this section, we give a few definitions and summarise results that we shall use later in our proof.

Given a graph G , we say $u, v \in V(G)$ are *connected* (in G) if there exists a path in G between u and v . The graph itself is said to be *connected* if any pair of vertices are connected. By extension, given a subgraph H of G , we say H is *connected* if, given any pair $u, v \in V(H)$, there exists a path in H between u and v and say that H is *effectively-connected* if, given any pair $u, v \in V(H)$, there exists a path in G between u and v .

A *connected-component* of a graph G is a maximal connected subgraph. A subgraph of H , a subgraph of G , is an *effectively-connected-component* or *effective-component* of H if it is a maximal effectively-connected subgraph of H . Thus the effective-components of H are restrictions of the components of G to H .

Given a multicoloured graph G , we say that two vertices u and v belong to the same *monochromatic component* of G if they belong to the same component of G_i for some i . Given a subgraph H of a multicoloured graph G , we say that two vertices u and v belong to the same *monochromatic effective-component* of H if they belong to the same effective-component of G_i for some i . We can thus talk about, for instance, the *red components* of a graph G or the *red effective-components* of a subgraph H of G .

We now summarise the results that we shall use later in our proofs. We begin with Dirac's Theorem, which gives us a minimum-degree condition for Hamiltonicity:

Theorem 2.1 (Dirac's Theorem [Dir52]). *If G is a graph on $n \geq 3$ vertices such that every vertex has degree at least $\frac{1}{2}n$, then G is Hamiltonian, that is, G contains a cycle of length exactly n .*

We also make use of the theorem of Erdős and Gallai:

Theorem 2.2 ([EG59]). *Any graph on K vertices with at least $\frac{1}{2}(m-1)(K-1) + 1$ edges, where $3 \leq m \leq K$, contains a cycle of length at least m .*

Observing that a cycle on m vertices contains a connected-matching on at least $m-1$ vertices, the following is an immediate consequence of the above.

Corollary 2.3. *For any graph G on K vertices and any m such that $3 \leq m \leq K$, if the average degree $d(G)$ is at least m , then G contains a connected-matching on at least m vertices.*

The following decomposition lemma of Figaj and Łuczak [FL07b] also follows from the theorem of Erdős and Gallai and is crucial in establishing the structure of a graph not containing large connected-matchings of the appropriate parities:

Lemma 2.4 ([FL07b, Lemma 9]). *For any graph G on K vertices and any m such that $3 \leq m \leq K$, if no odd component of G contains a matching on at least m vertices, then there exists a partition $V = V' \cup V''$ such that*

- (i) $G[V']$ is bipartite;
- (ii) every component of $G'' = G[V'']$ is odd;
- (iii) $G[V'']$ has at most $\frac{1}{2}m|V(G'')|$ edges; and
- (iv) there are no edges in $G[V', V'']$.

We recall two more results of Figaj and Łuczak. The first, is the main technical result from [FL07a]. The second from [FL07b], allows us to deal with graphs with a *hole*, that is, a subset $W \subseteq V(G)$ such that no edge of G lies inside W . Note that both of these results can be immediately extended to multicoloured graphs:

Lemma 2.5 ([FL07a, Lemma 8]). *For every $\alpha_1, \alpha_2, \alpha_3 > 0$ and $0 < \eta < 0.002 \min\{\alpha_1^2, \alpha_2^2, \alpha_3^2\}$, there exists $k_{2.5} = k_{2.5}(\alpha_1, \alpha_2, \alpha_3, \eta)$ such that the following holds:*

For every $k > k_{2.5}$ and every $(1 - \eta^4)$ -complete graph G on

$$K \geq \frac{1}{2} \left(\alpha_1 + \alpha_2 + \alpha_3 + \max\{\alpha_1, \alpha_2, \alpha_3\} + 18\eta^{1/2} \right) k$$

vertices, for every three-colouring of the edges of G , there exists a colour $i \in \{1, 2, 3\}$ such that G_i contains a connected-matching on at least $(\alpha_i + \eta)k$ vertices.

Lemma 2.6 ([FL07b, Lemma 12]). *For every $\alpha, \beta > 0$, $v \geq 0$ and η such that $0 < \eta < 0.01 \min\{\alpha, \beta\}$, there exists $k_{2.6} = k_{2.6}(\alpha, \beta, v, \eta)$ such that, for every $k > k_{2.6}$, the following holds:*

Let $G = (V, E)$ be a graph obtained from a $(1 - \eta^4)$ -complete graph on

$$K \geq \frac{1}{2} \left(\alpha + \beta + \max\{2v, \alpha, \beta\} + 6\eta^{1/2} \right) k$$

vertices by removing all edges contained within a subset $W \subseteq V$ of size at most vk . Then, every two-multicolouring of the edges of G results in either a red connected-matching on at least $(\alpha + \eta)k$ vertices or a blue connected-matching on at least $(\beta + \eta)k$ vertices.

The following pair of lemmas allow us to find large connected-matchings in almost-complete bipartite graphs:

Lemma 2.7 ([FL07b, Lemma 10]). *Let $G = G[V_1, V_2]$ be a bipartite graph with bipartition (V_1, V_2) , where $|V_1| \geq |V_2|$, which has at least $(1 - \epsilon)|V_1||V_2|$ edges for some ϵ such that $0 < \epsilon < 0.01$. Then, G contains a connected-matching on at least $2(1 - 3\epsilon)|V_2|$ vertices.*

Notice that, if G is a $(1 - \epsilon)$ -complete bipartite graph with bipartition (V_1, V_2) , then we may immediately apply the above to find a large connected-matching in G .

Lemma 2.8. *Let $G = G[V_1, V_2]$ be a bipartite graph with bipartition (V_1, V_2) . If ℓ is a positive integer such that $|V_1| \geq |V_2| \geq \ell$ and G is a -almost-complete for some a such that $0 < a/\ell < 0.5$, then G contains a connected-matching on at least $2|V_2| - 2a$ vertices.*

Proof. Observe that G is $(1 - a/\ell)$ -complete. Therefore, since $a/\ell < 0.5$, G is connected. Thus, it suffices to find a matching of the required size. Suppose that we have found a matching with vertex set M such that $|M| = 2k$ for some $k < |V_2| - a$ and consider a vertex $v_2 \in V_2 \setminus M$. Since G is a -almost-complete, v_2 has at least $|V_1| - a$ neighbours in $|V_1|$ and thus at least one neighbour in $v_1 \in V_1 \setminus M$. Then, the edge v_1v_2 can be added to the matching and thus, by induction, we may obtain a matching on $2|V_2| - 2a$ vertices. \square

We also make use of the following lemma from [KSS09], which is an extension of the two-colour Ramsey result for even cycles and which allows us to find, in any almost-complete two-multicoloured graph on K vertices, either a large matching or a particular structure.

Lemma 2.9 ([KSS09]). *For every η such that $0 < \eta < 10^{-20}$, there exists $k_{2.9} = k_{2.9}(\eta)$ such that, for every $k > k_{2.9}$ and every $\alpha, \beta > 0$ such that $\alpha \geq \beta \geq 100\eta^{1/2}\alpha$, if $K > (\alpha + \frac{1}{2}\beta - \eta^{1/2}\beta)k$ and $G = (V, E)$ is a two-multicoloured $\beta\eta^2k$ -almost-complete graph on K vertices, then at least one of the following occurs:*

- (i) G contains a red connected-matching on at least $(1 + \eta^{1/2})\alpha k$ vertices;
- (ii) G contains a blue connected-matching on at least $(1 + \eta^{1/2})\beta k$ vertices;
- (iii) the vertices of G can be partitioned into three sets W, V', V'' such that

- (a) $|V'| < (1 + \eta^{1/2})\alpha k$, $|V''| \leq \frac{1}{2}(1 + \eta^{1/2})\beta k$, $|W| \leq \eta^{1/16}k$,
- (b) $G_1[V']$ is $(1 - \eta^{1/16})$ -complete and $G_2[V']$ is $\eta^{1/16}$ -sparse,
- (c) $G_2[V', V'']$ is $(1 - \eta^{1/16})$ -complete and $G_1[V', V'']$ is $\eta^{1/16}$ -sparse;
- (iv) we have $\beta > (1 - \eta^{1/8})\alpha$ and the vertices of G can be partitioned into sets W , V' and V'' such that
 - (a) $|V'| < (1 + \eta^{1/2})\beta k$, $|V''| \leq \frac{1}{2}(1 + \eta^{1/8})\alpha k$, $|W| \leq \eta^{1/16}k$,
 - (b) $G_2[V']$ is $(1 - \eta^{1/16})$ -complete and $G_1[V']$ is $\eta^{1/16}$ -sparse,
 - (c) $G_1[V', V'']$ is $(1 - \eta^{1/16})$ -complete and $G_2[V', V'']$ is $\eta^{1/16}$ -sparse.

Furthermore, if $\alpha + \frac{1}{2}\beta \geq 2(1 + \eta^{1/2})\beta$, then we can replace (i) with

- (i') G contains a red odd connected-matching on $(1 + \eta^{1/2})\alpha k$ vertices.

We also make use of the following corollary of Lemma 2.9:

Corollary 2.10. *For every $0 < \epsilon < 10^{-12}$, there exists $k_{2.10} = k_{2.10}(\epsilon)$ such that, for every $k \geq k_{2.10}$, if $K > (1 - \epsilon)k$ and $G = (V, E)$ is a two-multicoloured $\frac{27}{8}\epsilon^4 k$ -almost-complete graph, then G contains at least one of the following:*

- (i) a red connected-matching on $(\frac{2}{3} - 7\epsilon^{1/8})k$ vertices;
- (ii) a blue connected-matching on $(\frac{2}{3} - 7\epsilon^{1/8})k$ vertices.

Proof. See [Fer15a]. □

It is a well-known fact that either a graph is connected or its complement is. The following three results are simple extensions of this fact for two-coloured almost-complete graphs, all of which can be immediately extended to two-multicoloured almost-complete graphs.

Lemma 2.11. *For every η such that $0 < \eta < 1/3$ and every $K \geq 1/\eta$, if $G = (V, E)$ is a two-coloured $(1 - \eta)$ -complete graph on K vertices and F is its largest monochromatic component, then $|F| \geq (1 - 3\eta)K$.*

Proof. See [Fer15a]. □

The following lemmas form analogues of the above, the first concerns the structure of two-coloured almost-complete graphs with one hole and the second concerns the structure of two-coloured almost-complete graphs with two holes, that is, bipartite graphs.

Lemma 2.12. *For every η such that $0 < \eta < 1/20$ and every $K \geq 1/\eta$, the following holds. For W , any subset of V such that $|W|, |V \setminus W| \geq 4\eta^{1/2}K$, let $G_W = (V, E)$ be a two-coloured graph obtained from G , a $(1 - \eta)$ -complete graph on K vertices with vertex set V by removing all edges contained entirely within W . Let F be the largest monochromatic component of G_W and define the following two sets:*

$$W_r = \{w \in W : w \text{ has red edges to all but at most } 3\eta^{1/2}K \text{ vertices in } V \setminus W\};$$

$$W_b = \{w \in W : w \text{ has blue edges to all but at most } 3\eta^{1/2}K \text{ vertices in } V \setminus W\}.$$

Then, at least one of the following holds:

- (i) $|F| \geq (1 - 2\eta^{1/2})K$; and
- (ii) $|W_r|, |W_b| > 0$.

Proof. See [Fer15a]. □

Lemma 2.13. *For every η such that $0 < \eta < 0.1$ and $K \geq 2/\eta$, the following holds: Suppose $G = (V, E)$ is a two-multicoloured graph obtained from an $(1 - \eta)$ -complete graph on K vertices with $V = A \cup B$ and $|A|, |B| \geq 6\eta K$ by removing all edges contained completely within A and all edges contained completely within B . Let F be the largest monochromatic component of G and define the following sets:*

$$\begin{aligned} A_r &= \{a \in A : a \text{ has red edges to all but at most } 4\eta K \text{ vertices in } B\}; \\ A_b &= \{a \in A : a \text{ has blue edges to all but at most } 4\eta K \text{ vertices in } B\}; \\ B_r &= \{b \in B : b \text{ has red edges to all but at most } 4\eta K \text{ vertices in } A\}; \\ B_b &= \{b \in B : b \text{ has blue edges to all but at most } 4\eta K \text{ vertices in } A\}. \end{aligned}$$

Then, at least one of the following occurs:

- (i) $|F| \geq (1 - 7\eta)K$;
- (ii) A, B can be partitioned into $A_1 \cup A_2, B_1 \cup B_2$ such that $|A_1|, |A_2|, |B_1|, |B_2| \geq 3\eta K$ and all edges present between A_i and B_j are red for $i = j$, blue for $i \neq j$;
- (iii) $|A_r|, |A_b| > 0$;
- (iv) $|B_r|, |B_b| > 0$.

Proof. Suppose $|F| < (1 - 7\eta)K$. Then, without loss of generality, A can be partitioned into $A_1 \cup A_2$, with $|A_1|, |A_2| \geq 3\eta K$, such that A_1 and A_2 are in different red components. Then, there exists no triple (a_1, b, a_2) with $a_1 \in A_1, b \in B, a_2 \in A_2$ and both a_1b and ba_2 coloured red. Thus, we may partition B into $B_1 \cup B_2$ such that there are no red edges present in $G[A_1, B_1]$ or $G[A_2, B_2]$.

Since G is $(1 - \eta)$ -complete, given any subsets $A' \subseteq A, B' \subseteq B$ every vertex in A' has degree at least $|B'| - \eta K$ in $G[A', B']$ and every vertex in B' has degree at least $|A'| - \eta K$ in $G[A', B']$. Thus, if $|B_1|, |B_2| \geq 3\eta K$, $G[A_1, B_1]$ and $G[A_2, B_2]$ each have a single blue component. Therefore, there can be no blue edges present in $G[A_1, B_2]$ or $G[A_2, B_1]$, giving rise to case (ii).

Thus, without loss of generality, we may assume that $|B_1| < 3\eta K$. Then, every vertex in A_2 is a vertex of A_b , in which case either every vertex $a \in A_1$ has a blue edge to B , leading to case (i), or there exists some $a \in A_1$ such that $a \in A_r$, giving rise to case (iii), thus completing the proof.

Note that exchanging the roles of A and B above leads to case (iv) in place of case (iii). □

3 Proof of the stability result – Part II

In [Fer15a, Section 7], we proved Theorem B in the case that $\alpha_1 \geq \alpha_3$, here we consider the case when $\alpha_3 \geq \alpha_1 \geq \alpha_2$. We wish to prove that any three-multicoloured $(1 - \eta^4)$ -complete graph on

slightly fewer than $\max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\}k$ vertices will have a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or will have a particular coloured structure.

Thus, given $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_3 \geq \alpha_1 \geq \alpha_2$, we set

$$c = \max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\},$$

choose

$$\eta < \eta_{B2} = \min \left\{ \frac{1}{10^5}, \frac{\alpha_2}{10^{24}}, \left(\frac{\alpha_2}{100} \right)^8, \left(\frac{\alpha_2}{1200\alpha_1} \right)^2 \right\}$$

and consider $G = (V, E)$, a $(1 - \eta^4)$ -complete graph on $K \geq 72/\eta$ vertices, where

$$(c - \eta)k \leq K \leq (c - \frac{\eta}{2})k$$

for some integer $k > k_{B2}$, where $k_{B2} = k_{B2}(\alpha_1, \alpha_2, \alpha_3, \eta)$ will be defined implicitly during the course of the proof, in that, on a finite number of occasions, we will need to bound k below in order to apply results from Section 2.

Note that, since $\alpha_3 \geq \alpha_1 \geq \alpha_2$, the largest forbidden connected-matching is green and odd. By scaling, we may assume that $2 \geq \alpha_3 \geq 1 \geq \alpha_1 \geq \alpha_2$. Thus, G is $3\eta^4 k$ -almost-complete, as is $G[X]$, for any $X \subset V$. We begin by noting that we can use Lemma 2.5 to obtain either a red connected-matching on $\alpha_1 k$ vertices, a blue connected-matching on $\alpha_2 k$ vertices or a green connected-matching of almost the required size. Note, however, that this green connected-matching need not be odd. Indeed, the graph has

$$|V| = vk \geq (c - \eta)k = (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + (v - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 - 9\eta^{1/2}) + 9\eta^{1/2})k$$

vertices and, since $\alpha_1 \geq \alpha_2$ and $\eta \leq (\frac{\alpha_2}{20})^2$, we have

$$\begin{aligned} (v - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 - 9\eta^{1/2}) &\geq (c - \eta - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 - 9\eta^{1/2}) \\ &\geq \max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\} - \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_2 - 10\eta^{1/2} \\ &\geq \max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2} \geq \alpha_1 \geq \alpha_2. \end{aligned}$$

Thus, since $\eta \leq 0.002 \min\{\alpha_1^2, \alpha_2^2, \alpha_3^2\}$, by Lemma 2.5, provided $k \geq k_{2.5}(\alpha_1, \alpha_2, v - \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 9\eta^{1/2}, \eta)$, G contains either a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices or a green connected-matching on at least

$$|V| - (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 9\eta^{1/2})k \geq (\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \quad (1)$$

vertices. Lemma 2.4 gives a decomposition of the green-graph G_3 into its bipartite and non-bipartite parts and in doing so gives a decomposition of the vertices of G into $X \cup Y \cup W$ such that there are no green edges between $X \cup Y$ and W or within X or Y . Choosing such a decomposition which maximises $|X \cup Y|$, results in $G_3[X \cup Y]$ being the union of the bipartite green components of G and $G_3[W]$ being the union of the non-bipartite green components of G . In what follows, we consider the vertices of G to have been thus partitioned. We will also assume that $|X| \geq |Y|$ and will write \hat{V} for $X \cup W$ and w for $|W|/k$. By (1), we may assume that the largest green connected-matching in G spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and distinguish three cases:

- (C) the largest green connected-matching F is not odd and $w \geq 7\eta^{1/2}$;
- (D) the largest green connected-matching F is not odd and $w \leq 7\eta^{1/2}$;
- (E) the largest green connected-matching F is odd.

Within each case, we will, when necessary, distinguish between the two possible forms taken by c , that is, between $c = 2\alpha_1 + \alpha_2$ and $c = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3$:

- (II †) The first possibility arises only when $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, in which case we may assume that F spans at least $(\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2})k$ vertices.
- (II ‡) The second possibility arises only when $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, in which case we may assume that that F spans at least $(\alpha_3 - 10\eta^{1/2})k$ vertices.

Case C: Largest green connected-matching is not odd and $w \geq 7\eta^{1/2}$.

Suppose that we have $c = 2\alpha_1 + \alpha_2$. Then F spans at least $(\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2})k$ vertices and is assumed to not be contained in an odd component of G . Thus, by the decomposition, we have

$$|X| \geq |Y| \geq (\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 - 5\eta^{1/2})k, \quad (2a)$$

$$|X| \geq \frac{K - |W|}{2} \geq (\alpha_1 + \frac{1}{2}\alpha_2 - \frac{1}{2}\eta - \frac{1}{2}w)k, \quad (2b)$$

$$|W| = K - |X| - |Y| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 10\eta^{1/2})k. \quad (2c)$$

From (2a) and (2b) we obtain

$$|\widehat{V}| = |X| + |W| \geq (\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 + w - 5\eta^{1/2})k, \quad (3a)$$

$$|\widehat{V}| = |X| + |W| \geq (\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}w - \frac{1}{2}\eta)k. \quad (3b)$$

Now, suppose that $|\widehat{V}| \geq \frac{1}{2}(\alpha_1 + \alpha_2 + \max\{2w, \alpha_1, \alpha_2\} + 6\eta^{1/2})k$. In that case, since $\eta \leq 0.01\alpha_2$, provided $k > k_{2.6}(\alpha_1, \alpha_2, w, \eta)$, we may apply Lemma 2.6 to obtain either a red connected-matching on $\alpha_1 k$ vertices or a blue connected-matching on $\alpha_2 k$ vertices. Therefore, we may assume that

$$|\widehat{V}| \leq \frac{1}{2}(\alpha_1 + \alpha_2 + \max\{2w, \alpha_1, \alpha_2\} + 6\eta^{1/2})k. \quad (3c)$$

If $w \leq \frac{1}{2}\alpha_1$, then together (3b) and (3c) contradict our assumption that $w \geq 7\eta^{1/2}$. Thus, we may assume that $w \geq \frac{1}{2}\alpha_1$. In that case, (3a) and (3c) give

$$(\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 + w - 5\eta^{1/2})k \leq |\widehat{V}| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + w + 3\eta^{1/2})k,$$

which together with (2a) and (2c) gives

$$\begin{aligned} (\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 - 5\eta^{1/2})k &\leq |X| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 - 5\eta^{1/2})k &\leq |Y| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\alpha_1 - 7\eta^{1/2})k &\leq |W| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 10\eta^{1/2})k. \end{aligned}$$

The condition for $|X|$ above gives a contradiction unless $\alpha_1 \leq \alpha_2 + 32\eta^{1/2}$. So, recalling that $\alpha_2 \leq \alpha_1$, we may obtain

$$\left. \begin{aligned} (\alpha_1 - 13\eta^{1/2})k &\leq |X| \leq (\alpha_1 + 3\eta^{1/2})k, \\ (\alpha_1 - 13\eta^{1/2})k &\leq |Y| \leq (\alpha_1 + 3\eta^{1/2})k, \\ (\alpha_1 - 7\eta^{1/2})k &\leq |W| \leq (\alpha_1 + 10\eta^{1/2})k. \end{aligned} \right\} \quad (4)$$

Suppose instead that $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$. Then, by (1), F spans at least $(\alpha_3 - 10\eta^{1/2})k$ vertices. Recall that we assume that F is not contained in an odd component of G , thus, by the decomposition, we have

$$|X| \geq |Y| \geq (\tfrac{1}{2}\alpha_3 - 5\eta^{1/2})k, \quad (5a)$$

$$|X| \geq \frac{K - |W|}{2} \geq (\alpha_1 + \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\eta - \tfrac{1}{2}w)k, \quad (5b)$$

$$|W| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2})k. \quad (5c)$$

From (5a) and (5b), we obtain

$$|\widehat{V}| = |X| + |W| \geq (\tfrac{1}{2}\alpha_3 + w - 5\eta^{1/2})k, \quad (6a)$$

$$|\widehat{V}| = |X| + |W| \geq (\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}w - \tfrac{1}{2}\eta)k. \quad (6b)$$

Again, we may assume that

$$|\widehat{V}| \leq \tfrac{1}{2} \left(\alpha_1 + \tfrac{1}{2}\alpha_2 + \max\{2w, \alpha_1, \alpha_2\} + 6\eta^{1/2} \right) k \quad (6c)$$

since, otherwise, we may apply Lemma 2.6 to obtain either a red connected-matching on $\alpha_1 k$ vertices or a blue connected-matching on $\alpha_2 k$ vertices. Again, we may assume that $w \geq \frac{1}{2}\alpha_1$ since, otherwise, together (6b) and (6c) contradict our assumption that $w \geq 7\eta^{1/2}k$. Then, (6a) and (6c) give

$$(\tfrac{1}{2}\alpha_3 + w - 5\eta^{1/2})k \leq |\widehat{V}| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + w + 3\eta^{1/2})k,$$

which, together with (5a) and (5c) gives

$$\begin{aligned} (\tfrac{1}{2}\alpha_3 - 5\eta^{1/2})k &\leq |X| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_3 - 5\eta^{1/2})k &\leq |Y| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\alpha_3 - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 - 7\eta^{1/2})k &\leq |W| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2})k. \end{aligned}$$

Then, recalling that $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$ and that $\alpha_2 \leq \alpha_1$, in order to avoid a contradiction, we have $\alpha_1 \leq \alpha_2 + 32\eta^{1/2}$ and obtain

$$\left. \begin{aligned} (\alpha_1 - 13\eta^{1/2})k &\leq |X| \leq (\alpha_1 + 3\eta^{1/2})k, \\ (\alpha_1 - 13\eta^{1/2})k &\leq |Y| \leq (\alpha_1 + 3\eta^{1/2})k, \\ (\alpha_1 - 7\eta^{1/2})k &\leq |W| \leq (\alpha_1 + 10\eta^{1/2})k. \end{aligned} \right\} \quad (7)$$

Considering (4) and (7), we see that we have obtained the same set of bounds irrespective of the form

taken by c . Thus, in what follows, we consider both possibilities together.

Recall that, under the decomposition, there are no green edges contained within X or Y . Then, since G is $3\eta^4 k$ -almost-complete, provided $k > k_{2.10}(\eta)$, we may apply Corollary 2.10 to each of $G[X]$ and $G[Y]$, thus finding that each contains a monochromatic connected-matching on at least $(\frac{2}{3}\alpha_1 - 8\eta^{1/8})k$ vertices. Thus, provided $\eta < (\alpha_1/120)^8$, we may assume that each of X and Y contain a monochromatic connected-matching on at least $\frac{3}{5}\alpha_1 k$ vertices. Referring to these matchings as $M_1 \subseteq G[X]$ and $M_2 \subseteq G[Y]$, we consider three subcases:

- (i) M_1 and M_2 are both red;
- (ii) M_1 and M_2 are both blue;
- (iii) M_1 and M_2 are different colours.

The proof in the first subcase is identical to that of Case B.i in [Fer15a], the proof in the second subcase is identical to that of Case B.ii in [Fer15a] and the proof in the third subcase is identical to that of Case B.iii in [Fer15a] with the overall result being that G contains either a red connected-matching on $\alpha_1 k$ vertices or a blue connected-matching on $\alpha_2 k$ vertices.

Case D: Largest green connected-matching is not odd and $w \leq 7\eta^{1/2}$.

Suppose that $c = 2\alpha_2 + \alpha_2$. Then, since $\eta \leq \eta_{B2}$, provided $k \geq k_{2.6}(\alpha_1, \alpha_2, w, \eta)$, we obtain bounds on the sizes of X and Y as follows:

$$\begin{aligned} (\alpha_1 + \frac{1}{2}\alpha_2 - 4\eta^{1/2})k &\leq |X| \leq (\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\alpha_1 + \frac{1}{2}\alpha_2 - 12\eta^{1/2})k &\leq |Y| \leq (\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k. \end{aligned}$$

Suppose instead that $c = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3$. Then, since $\eta \leq \eta_{B2}$, provided $k \geq k_{2.6}(\alpha_1, \alpha_2, w, \eta)$, we obtain bounds on the sizes of X and Y as follows:

$$\begin{aligned} (\frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 + \frac{1}{2}\alpha_3 - 4\eta^{1/2})k &\leq |X| \leq (\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k, & (8a) \\ (\alpha_3 - \frac{1}{2}\alpha_1 - 12\eta^{1/2})k &\leq |Y| \leq (\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k. & (8b) \end{aligned}$$

Note that the inequalities in (8a) give a contradiction unless $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 14\eta^{1/2}$. Since $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, we obtain

$$\begin{aligned} (\alpha_1 + \frac{1}{2}\alpha_2 - 4\eta^{1/2})k &\leq |X| \leq (\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\alpha_1 + \frac{1}{2}\alpha_2 - 12\eta^{1/2})k &\leq |Y| \leq (\alpha_1 + \frac{1}{2}\alpha_2 + 3\eta^{1/2})k. \end{aligned}$$

Then, in either case, since $\eta \leq \eta_{B2}$, provided $k > k_{2.9}(\eta^{1/2})$, we may apply Lemma 2.9 (with $\alpha = \alpha_1, \beta = \alpha_2$) to each of X and Y to find that each contains a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices or has a structure belonging to one of the following classes as a subgraph:

$$\begin{aligned} \mathcal{H}_1 &= \mathcal{H}\left((\alpha_1 - 2\eta^{1/32})k, (\frac{1}{2}\alpha_2 - 2\eta^{1/32})k, 3\eta^4 k, \eta^{1/32}, \text{red, blue}\right); \\ \mathcal{H}_2 &= \mathcal{H}\left((\alpha_2 - 2\eta^{1/32})k, (\frac{1}{2}\alpha_1 - 2\eta^{1/32})k, 3\eta^4 k, \eta^{1/32}, \text{blue, red}\right), \end{aligned}$$

with the latter case occurring only if $\alpha_2 \geq \alpha_1 - \eta^{1/16}$.

Case E: Largest green connected-matching is odd.

Recall, from (1), that F , the largest green connected-matching in G , spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices. We now consider the case when this connected-matching is contained in an odd component of G .

Thus far, we have made extensive use of the decomposition of Figaj and Łuczak described in Lemma 2.4. However, in this case, it is necessary to consider an alternative (and somewhat more complicated) decomposition:

We begin by partitioning the vertices of G into $L \cup P \cup Q$ as follows. We let L be the vertex set of F . Then, for each $v \in V \setminus L$, if there exists a green edge between v and L , we assign v to P ; otherwise, we assign v to Q .

Suppose there exists a green edge mn in F and distinct vertices $p_1, p_2 \in P$ such that mp_1 and np_2 are both coloured green. Then, we can replace mn with mp_1 and np_2 , contradicting the maximality of F . Thus, after discarding at most one edge from $G[L, P]$ for each edge of F , we may assume that given an edge uv in the matching, at most one of u or v has a green edge to P . We may therefore partition L into $M \cup N$ such that each edge of the matching belongs to $G[M, N]$ and there are no green edges in $G[N, P]$. Observe also that, by maximality of F , there can be no green edges within $G[P]$ or $G[P, Q]$.

In summary, we have a partition $M \cup N \cup P \cup Q$ such that

- (E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;
- (E2) every vertex in P has a green edge to M ;
- (E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$.

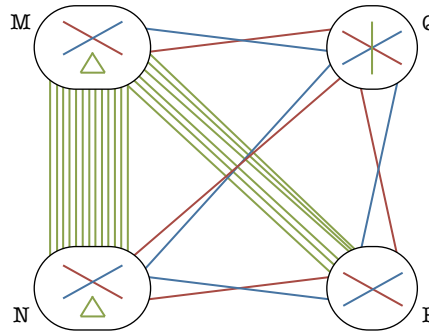


Figure 3: Decomposition into $M \cup N \cup P \cup Q$.

Note that, since G was assumed to be $(1 - \eta^4)$ -complete and also $3\eta^4 k$ -almost-complete, having discarded the green edges described above, provided $k \geq 1/\eta^4$, we may now assume that the (new) graph is $(1 - \frac{3}{2}\eta^4)$ -complete and also $4\eta^4 k$ -almost-complete. In what follows, on a number of occasions, we will discard vertices from $M \cup N \cup P \cup Q$ but will continue to refer the parts of the partition as M, N, P and Q . The discarded vertices remain in the graph and will be considered later. We need to take care to account of this when considering the sizes of $V(G), M, N, P, Q$, etc.

Recalling (II \dagger), in the case that $c = 2\alpha_1 + \alpha_2$, we have $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$. Then, since $V(F) = M \cup N$ and $|M| + |N| + |P| + |Q| = K$, we have

$$(\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3 k, \quad (9a)$$

$$(2\alpha_1 + \alpha_2 - \alpha_3 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (9b)$$

The inequalities in (9a) give a contradiction unless $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}$. Then, since $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, we may re-write (9b) as

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (9b')$$

Recalling (II \ddagger), in the case that $c = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3$, we have $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$. Then, since $V(F) = M \cup N$ and $|M| + |N| + |P| + |Q| = K$, we have

$$(\frac{1}{2}\alpha_3 - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3 k, \quad (10a)$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (10b)$$

We will proceed considering the two possible situations together, assuming that

$$\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}. \quad (11)$$

Comparing (9a) to (10a) and (9b') to (10b), we will assume that

$$(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3 k, \quad (E4a)$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (E4b)$$

and distinguish between three possibilities:

- (i) $|P| \leq 95\eta^{1/2}k$;
- (ii) $|Q| \leq 95\eta^{1/2}k$;
- (iii) $|P|, |Q| \geq 95\eta^{1/2}k$.

Case E.i: $|P| \leq 95\eta^{1/2}k$.

In this case, we disregard P , and, recalling that $L = M \cup N$, consider $G[L \cup Q]$. From (E4a) and (E4b), we have

$$(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \leq |L| \leq \alpha_3 k, \quad (E4a')$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 96\eta^{1/2})k \leq |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (E4b')$$

By (E3), we know that all edges in $G[L, Q]$ are coloured red or blue.

Observe that, provided $\eta < (1/200)^2$, we have

$$|L| + |Q| \geq \left(\max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\} - 106\eta^{1/2} \right) k \geq \frac{3}{4}K.$$

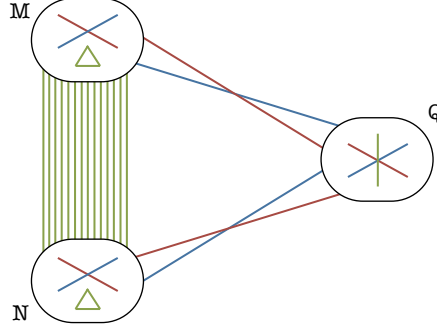


Figure 4: Decomposition in Case E.i.

Thus, since G is $(1 - \frac{3}{2}\eta^4)$ -complete, $G[L \cup Q]$ is $(1 - 2\eta^4)$ -complete. Also, provided $\eta < 10^{-5}$, we have $|L|, |Q| \geq 18\eta^{1/2}(|L| + |Q|)$. Thus, since $2\eta^4 \leq 3\eta^{1/2}$, provided $K \geq 2/\eta^{1/2}$, we may apply Lemma 2.13, giving rise to four cases:

- (a) $G[L, Q]$ contains a monochromatic component on at least $(1 - 21\eta^{1/2})|L \cup Q| \geq |L \cup Q| - 63\eta^{1/2}k$ vertices;
- (b) L, Q can be partitioned into $L_1 \cup L_2, Q_1 \cup Q_2$ such that $|L_1|, |L_2|, |Q_1|, |Q_2| \geq 9\eta^{1/2}|L \cup Q| \geq 9\eta^{1/2}k$ and all edges present between L_i and Q_j are red for $i = j$, blue for $i \neq j$;
- (c) there exist vertices $v_r, v_b \in L$ such that v_r has red edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in Q and v_b has blue edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in Q ;
- (d) there exist vertices $v_r, v_b \in Q$ such that v_r has red edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in L and v_b has blue edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in L .

Case E.i.a: $G[L \cup Q]$ has a large monochromatic component.

Recall that we assume that F , the largest green connected-matching in G , spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and is contained in an odd component of G . We have a partition of $V(G)$ into $L \cup P \cup Q$ such that $|P| \leq 95\eta^{1/2}k$,

$$(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \leq |L| \leq \alpha_3 k, \quad (\text{E4a}')$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 96\eta^{1/2})k \leq |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b}')$$

Recalling that $L = M \cup N$, by (E3), all edges present in $G[L, Q]$ are coloured red or blue. Additionally, in this case, we assume that $G[L, Q]$ contains a monochromatic component on at least $|L \cup Q| - 63\eta^{1/2}k$ vertices. Suppose this large monochromatic component is red, then

(E5) $G[L, Q]$ has a red component on at least $|L \cup Q| - 63\eta^{1/2}k$ vertices.

We consider the largest red matching R in $G[L, Q]$ and, thus, partition L into $L_1 \cup L_2$ and Q into $Q_1 \cup Q_2$ where $L_1 = L \cap V(R)$, $L_2 = L \setminus L_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \setminus Q_1$. By maximality of R , all edges present in $G[L_2, Q_2]$ are coloured exclusively blue. Notice that, since $\eta \leq (\alpha_1/100)^2$, we have, by (E4a') and (E4b'), $|L| \geq |Q|$. Thus, since $|L_1| = |Q_1|$, we have $|L_2| \geq |Q_2|$ and so, in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices, by Lemma 2.8, we have $|Q_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$ and, therefore, also $|L_1| = |Q_1| = |Q| - |Q_2| \geq (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k$.

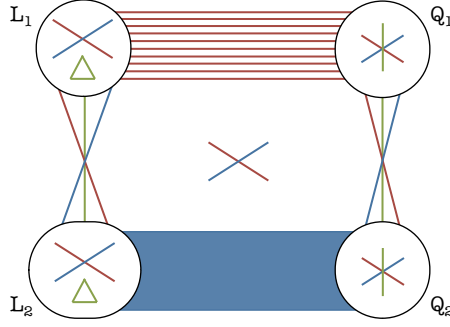


Figure 5: Decomposition into $L_1 \cup L_2 \cup Q_1 \cup Q_2$ in Case E.i.a.

Also, by (E5), in order to avoid having a red connected-matching on at least $\alpha_1 k$ vertices, we may assume that $|L_1| = |Q_1| \leq (\frac{1}{2}\alpha_1 + 64\eta^{1/2})k$. Finally, we have $|Q_2| = |Q| - |Q_1| \geq (\frac{1}{2}\alpha_2 - 160\eta^{1/2})k$.

In summary, we have $|L_1| = |Q_1|$,

$$(\frac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |Q_1| \leq (\frac{1}{2}\alpha_1 + 64\eta^{1/2})k, \quad (15a)$$

$$(\frac{1}{2}\alpha_2 - 160\eta^{1/2})k \leq |Q_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k. \quad (15b)$$

Note that, since $\eta \leq (\alpha_1/10000)^2$, by (E4a'), (E4b'), we have $|L| \geq |Q| + 3000\eta^{1/2}k$ and thus, since $|Q_1| = |L_1|$, also have

$$|L_2| \geq |Q_2| + 3000\eta^{1/2}k. \quad (15c)$$

Recalling that $|L_2| = |L| - |L_1| = |L| - |Q_1|$, since $\eta \leq (\alpha_1/10000)^2$, considering (E4a') and (15a), we also have

$$|L_2| \geq |Q_1| + 3000\eta^{1/2}k. \quad (15d)$$

Equations (15c) and (15d) are crucial to the argument that follows since they provide us with the spare vertices we will need in order to establish the coloured structure of G . In what follows, we will show that, after possibly discarding some vertices from each of L_1, L_2, Q_1 and Q_2 , we may assume that all edges present in $G[L, Q_1]$ are coloured exclusively red and that all edges present in $G[L, Q_2]$ are coloured exclusively blue. This is done in three steps, the first dealing with $G[L_2, Q_1]$, the second dealing with $G[L_1, Q_2]$ and the third dealing with $G[L_1, Q_1]$. Similar arguments will appear many times throughout the remainder of the proof of Theorem B. Note that, in what follows, we mostly omit floors and ceilings for the sake of clarity of presentation, we may do this since we are free to increase k where necessary.

Claim 3.1. *We may discard at most $842\eta^{1/2}k$ vertices from each of L_1 and Q_1 , at most $161\eta^{1/2}k$ vertices from L_2 and at most $260\eta^{1/2}k$ vertices from Q_2 such that all remaining in $G[L, Q_1]$ are coloured exclusively red and all edges present in $G[L, Q_2]$ are coloured exclusively blue.*

Proof. We begin by considering the blue graph. Observing that $G[L_2, Q_2]$ contains a blue connected-matching of size close to $\alpha_2 k$ and that there are ‘spare’ vertices in L_2 . Thus, we note that there can only be few blue edges in $G[L_2, Q_1]$. Indeed, suppose there exists a blue matching B_S on at least $322\eta^{1/2}k$ vertices in $G[L_2, Q_1]$.

Then, by (15d), letting $\tilde{L} = L_2 \setminus V(B_S)$, we have $|\tilde{L}| \geq |Q_2| \geq (\frac{1}{2}\alpha_2 - 160\eta^{1/2})k$. Thus, by Lemma 2.8,

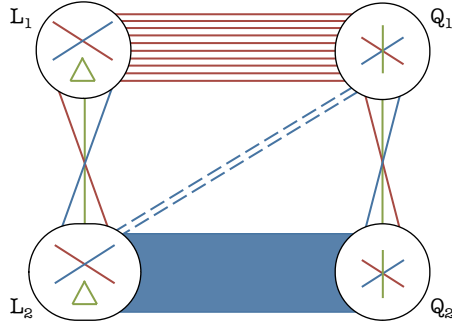


Figure 6: The blue matching B_S .

there exists a blue connected-matching B_L on at least $(\alpha_2 - 322\eta^{1/2})k$ vertices in $G[\tilde{L}, Q_2]$ which shares no vertices with B_S . Notice that, since $G_2[L_2, Q_2]$ is $4\eta^4 k$ -almost-complete, $L_2 \cup Q_2$ forms a single blue component in G and, thus, $B_S \cup B_L$ forms a blue connected-matching on $\alpha_2 k$ vertices. Therefore, no such matching as B_S can exist. So, after discarding at most $161\eta^{1/2}k$ vertices from each of Q_1 and L_2 , we may assume that all edges present in $G[L_2, Q_1]$ are coloured exclusively red.

In order to retain the equality $|L_1| = |Q_1|$ and the property that every vertex in L_1 belongs to an edge of R , we discard from L_1 each vertex whose R -mate in Q_1 has already been discarded. Recalling (15a)–(15d), we now have

$$\begin{aligned} |L_2| &\geq |Q_1| + 2800\eta^{1/2}k, & |L_1| = |Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 258\eta^{1/2})k, \\ |L_2| &\geq |Q_2| + 2800\eta^{1/2}k, & |Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 160\eta^{1/2})k. \end{aligned}$$

We now consider the red graph. Since all edges in $G[L_2, Q_1]$ are coloured exclusively red, any two vertices in Q_1 have a common red neighbour in L_2 . Thus, since every vertex in L_1 has a red neighbour in Q_1 , we know that $G[L_1 \cup Q_1]$ has a single effective red component. Suppose, then, that there exists a red matching R_S on at least $520\eta^{1/2}k$ vertices in $G[L_1, Q_2]$. Then, recalling that the matching R spans all the vertices of $G[L_1, Q_1]$, we may construct a red connected-matching on at least $\alpha_1 k$ vertices as follows.

Observe that there exists a set R^- of $260\eta^{1/2}k$ edges belonging to R such that $L_1 \cap V(R_S) = L_1 \cap V(R^-)$. Define $R^* = R \setminus R^-$ and $\tilde{Q} = Q_1 \cap V(R^-)$ and consider $G[L_2, \tilde{Q}]$. Since $|L_2|, |\tilde{Q}| \geq 260\eta^{1/2}k$ and $G[L_2, \tilde{Q}]$ is $4\eta^4 k$ -almost-complete, by Lemma 2.8, there exists a red connected-matching R_T on at least $518\eta^{1/2}k$ vertices in $G[L_2, \tilde{Q}]$. Then, $R^* \cup R_S \cup R_T$ is a red-connected-matching in $G[L_1, Q_1] \cup G[L_1, Q_2] \cup G[L_2, Q_1]$ on at least $2(|Q_1| - 261\eta^{1/2}k) + 520\eta^{1/2}k + 518\eta^{1/2}k \geq \alpha_1 k$ vertices.

Therefore, a matching such as R_S cannot exist. Thus, after discarding at most $260\eta^{1/2}k$ vertices from each of L_1 and Q_2 , we may assume that all edges present in $G[L_1, Q_2]$ are coloured exclusively blue. Note that, in order to retain the equality $|L_1| = |Q_1|$, we also discard from Q_1 each vertex whose R -mate in L_1 has already been discarded. After discarding vertices, we have

$$|L_1| = |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 518\eta^{1/2}), \quad |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 420\eta^{1/2}).$$

To complete the claim, we return to the blue graph. Since all edges present in $G[L, Q_2]$ are coloured exclusively blue and G is $4\eta^4 k$ -almost-complete, $L \cup Q_2$ forms a single blue component. Also, since

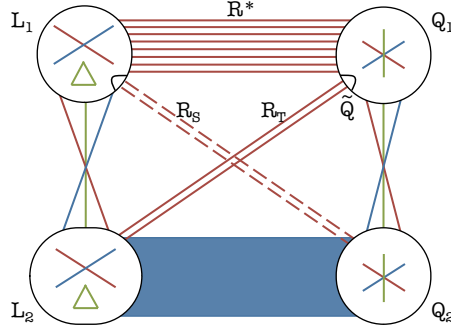


Figure 7: Construction of a red connected-matching on $\alpha_1 k$ vertices.

$|L_2|, |Q_2| \geq (\frac{1}{2}\alpha_2 - 420\eta^{1/2})k$, by Lemma 2.8, there exists a blue connected-matching B_2 on at least $(\alpha_2 - 842\eta^{1/2})k$ vertices in $G[L_2, Q_2]$. Thus, if there existed a blue matching B_1 on at least $842\eta^{1/2}k$ vertices in $G[L_1, Q_1]$, then $B_1 \cup B_2$ would form a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $421\eta^{1/2}k$ vertices from each of L_1, Q_1 , we may assume that all edges present in $G[L_1, Q_1]$ are coloured exclusively red. Thus completing the proof of the claim. \square

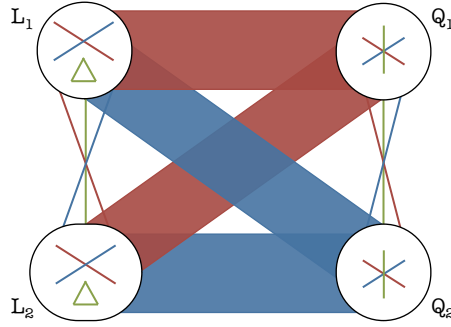


Figure 8: Colouring of $G[L, Q]$ after Claim 3.1.

Having proved Claim 3.1, we know that all edges present in $G[L, Q_1]$ are coloured exclusively red and that all edges present in $G[L, Q_2]$ are coloured exclusively blue. Additionally, we have

$$\left. \begin{aligned} |L_2| &\geq |Q_1| + 2800\eta^{1/2}k, & |L_1| = |Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 939\eta^{1/2})k, \\ |L_2| &\geq |Q_2| + 2800\eta^{1/2}k, & |Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 420\eta^{1/2})k. \end{aligned} \right\} \quad (16)$$

Now, suppose that there exists a red matching R^+ on $1880\eta^{1/2}k$ vertices in L . Then, by (16), we have $|L \setminus V(R^+)| \geq |Q_1| \geq (\frac{1}{2}\alpha_1 - 939\eta^{1/2})k$. So, by Lemma 2.8, there exists a red connected-matching on at least $(\alpha_1 - 1880\eta^{1/2})k$ vertices in $G[L \setminus V(R^+), Q_1]$, which can be augmented with edges from R^+ to give a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $1880\eta^{1/2}k$ vertices from L , we may assume that there are no red edges present in $G[L]$ and that we then have

$$|L| \geq |Q_1| + 900\eta^{1/2}k, \quad |L| \geq |Q_2| + 900\eta^{1/2}k.$$

Finally, suppose that there exists a blue connected-matching B^+ on $842\eta^{1/2}k$ vertices in L . Then, $|L \setminus V(B^+)| \geq |Q_2| \geq (\frac{1}{2}\alpha_2 - 420\eta^{1/2})k$ so, by Lemma 2.8, there exists a blue connected-matching on

at least $(\alpha_2 - 842\eta^{1/2})k$ vertices in $G[L \setminus V(B^+), Q_2]$, which can be augmented with edges from B^+ to give a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most a further $842\eta^{1/2}$ vertices from L , we may assume that all edges present in $G[L]$ are coloured exclusively green.

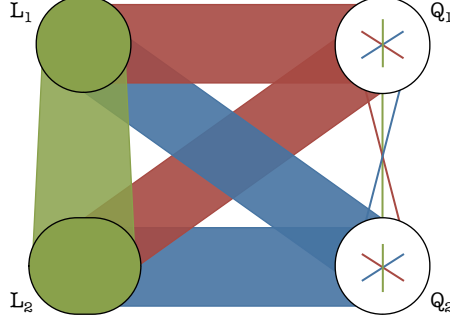


Figure 9: Final colouring in Case E.i.a.

In summary, having discarded at most $4828\eta^{1/2}k$ vertices, we have

$$|Q_1| \geq (\tfrac{1}{2}\alpha_1 - 939\eta^{1/2})k, \quad |Q_2| \geq (\tfrac{1}{2}\alpha_1 - 420\eta^{1/2})k, \quad |L| \geq (\tfrac{1}{2}\alpha_3 - 3750\eta^{1/2})k,$$

and know that all edges present in $G[Q_1, L]$ are coloured exclusively red, all edges present in $G[Q_2, L]$ are coloured exclusively blue and all edges present in $G[L]$ are coloured exclusively green. Thus, we have found, as a subgraph of G , a graph belonging to

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 1000\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 1000\eta^{1/2})k, (\alpha_3 - 4000\eta^{1/2})k, 4\eta^4 k\right).$$

At the beginning of Case E.i.a, we assumed that the largest monochromatic component in $G[L, Q]$ was red. If, instead, that monochromatic component is blue, then the proof proceeds exactly as above with the roles of red and blue exchanged and with α_1 and α_2 exchanged. The result is identical.

Case E.i.b: $L \cup Q$ has a non-trivial partition with ‘cross’ colouring.

Recall that we assume that F , the largest green connected-matching in G , spans at least $(\max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and is contained in an odd component of G . We have a partition of $V(G)$ into $L \cup P \cup Q$, such that $|P| \leq 95\eta^{1/2}k$,

$$(\max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \leq |L| \leq \alpha_3 k, \tag{E4a'}$$

$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 96\eta^{1/2})k \leq |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{E4b'}$$

Additionally, in this subcase, we assume that L and Q can be partitioned into $L_1 \cup L_2$ and $Q_1 \cup Q_2$ such that $|L_1|, |L_2|, |Q_1|, |Q_2| \geq 9\eta^{1/2}k$ and all edges present in $G[L_i, Q_j]$ are coloured exclusively red for $i = j$, and exclusively blue for $i \neq j$. Then, by Lemma 2.8, there exist red connected-matchings

$$M_{11} \text{ on at least } 2(\min\{|L_1|, |Q_1|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_1, Q_1], \tag{18a}$$

$$M_{22} \text{ on at least } 2(\min\{|L_2|, |Q_2|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_2, Q_2], \tag{18b}$$

and blue connected-matchings

$$M_{12} \text{ on at least } 2(\min\{|L_1|, |Q_2|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_1, Q_2], \quad (18c)$$

$$M_{21} \text{ on at least } 2(\min\{|L_2|, |Q_1|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_2, Q_1]. \quad (18d)$$

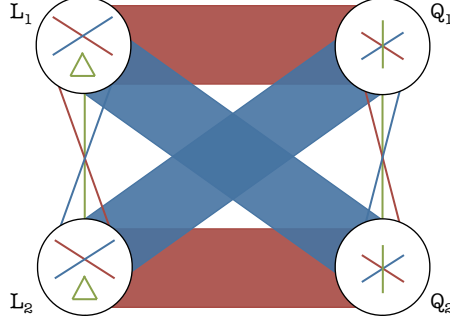


Figure 10: Decomposition into $L_1 \cup L_2 \cup Q_1 \cup Q_2$ in Case E.i.b.

Thus, in order to avoid a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices, we may assume that $|V(M_{11})|, |V(M_{22})| \leq \alpha_1 k$ and $|V(M_{12})|, |V(M_{21})| \leq \alpha_2 k$. Thus, (18a)–(18d), above can be used to obtain bounds on the sizes of L_1, L_2, Q_1 and Q_2 as follows:

$$\text{Since } |V(M_{11})|, |V(M_{22})| \leq \alpha_1 k, \text{ we have} \quad \min\{|L_1|, |Q_1|\} \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \quad (19a)$$

$$\text{and} \quad \min\{|L_2|, |Q_2|\} \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k. \quad (19b)$$

$$\text{Since } |V(M_{12})|, |V(M_{21})| \leq \alpha_2 k, \text{ we have} \quad \min\{|L_1|, |Q_2|\} \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \quad (19c)$$

$$\text{and} \quad \min\{|L_2|, |Q_1|\} \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k. \quad (19d)$$

Observe that, since $\eta \leq (\alpha_1/15)^2$, by (E4a') and (E4b'), we have

$$\begin{aligned} |L_1| + |L_2| = |L| &\geq (\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2})k \\ &\geq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k + (\alpha_1 - 15\eta^{1/2})k \geq |Q| = |Q_1| + |Q_2|. \end{aligned}$$

Thus, it is not possible to have, for instance, $|Q_1|, |Q_2| \geq |L_1|, |L_2|$. Without loss of generality, we therefore consider four possibilities

- (i) $|L_1|, |L_2| \geq |Q_1|, |Q_2|$;
- (ii) $|L_1| \geq |Q_1|, |Q_2| \geq |L_2|$;
- (iii) $|L_1| \geq |Q_i| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$;
- (iv) $|Q_i| \geq |L_1| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$.

Case E.i.b.i: $|L_1|, |L_2| \geq |Q_1|, |Q_2|$.

In this case, by (19c) and (19d), we may assume that $|Q_1|, |Q_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k$ and, consequently, by (E4b'), we have

$$|Q_1|, |Q_2| \geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k.$$

Thus, by (18a)–(18d), we have red connected-matchings M_{11}, M_{22} , each on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices and blue connected-matchings M_{12}, M_{21} , each on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices. (Also, in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices, we may assume that $\alpha_1 \leq \alpha_2 + 196\eta^{1/2}$.) Notice that there can be no red edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$, since any such edge would mean M_{11} and M_{22} being in the same red-component and so $M_{11} \cup M_{22}$ would form a red connected-matching on at least $\alpha_1 k$ vertices. Similarly, there can be no blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$. Therefore, all edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$ are coloured exclusively green.

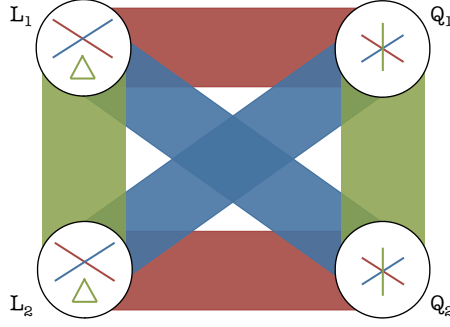


Figure 11: Colouring of $G[L_1 \cup L_2 \cup Q_1 \cup Q_2]$ in Case E.i.b.i.

Thus, we have found, as a subgraph of G , a graph belonging to

$$\begin{aligned} \mathcal{K}^* \left(\left(\frac{1}{2}\alpha_1 - 97\eta^{1/2} \right)k, \left(\frac{1}{2}\alpha_1 - 97\eta^{1/2} \right)k, \left(\frac{1}{2}\alpha_1 - 97\eta^{1/2} \right)k, \right. \\ \left. \left(\frac{1}{2}\alpha_1 - 97\eta^{1/2} \right)k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k \right) \subseteq \mathcal{K}_1^* \cup \mathcal{K}_2^*. \end{aligned}$$

Case E.i.b.ii: $|L_1| \geq |Q_1|, |Q_2| \geq |L_2|$.

In this case, by (19a) and (19c), we have

$$|Q_1| \leq \left(\frac{1}{2}\alpha_1 + \eta^{1/2} \right)k, \quad |Q_2| \leq \left(\frac{1}{2}\alpha_2 + \eta^{1/2} \right)k, \quad (20)$$

and so, by (E4b'),

$$|Q_1| \geq \left(\frac{1}{2}\alpha_1 - 97\eta^{1/2} \right)k, \quad |Q_2| \geq \left(\frac{1}{2}\alpha_2 - 97\eta^{1/2} \right)k. \quad (21)$$

Thus, by (18a), we have a red connected-matching M_{11} on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices in $G[L_1, Q_1]$ and, by (18c), have a blue connected-matching M_{12} on at least $(\alpha_2 - 196\eta^{1/2})k$ vertices in $G[L_1, Q_2]$.

Suppose, then, that $|L_2| \geq 100\eta^{1/2}k$. Then, by Lemma 2.8, there exists a red connected-matching R_S on at least $198\eta^{1/2}k$ vertices in $G[L_2, Q_2]$ and a blue connected-matching B_S on at least $198\eta^{1/2}k$ vertices in $G[L_2, Q_1]$. Thus, there can be no red edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$, since otherwise M_{11} and R_S would belong to the same red component and together span at least $\alpha_1 k$ vertices. Likewise, there can be no blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$, since then M_{12} and B_S would then belong to the same blue component and together span at least $\alpha_2 k$ vertices. Therefore, we have, as a subgraph of G , a graph belonging to

$$\mathcal{K}^*\left(\left(\frac{1}{2}\alpha_1 - 97\eta^{1/2}\right)k, \left(\frac{1}{2}\alpha_2 - 97\eta^{1/2}\right)k, \left(\alpha_3 - \frac{1}{2}\alpha_2 - 12\eta^{1/2}\right)k, 100\eta^{1/2}k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4k\right) \subset \mathcal{K}_2^*.$$

Thus, we may assume that $|L_2| \leq 100\eta^{1/2}k$, in which case we have

$$|L_1| \geq (\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 110\eta^{1/2})k \quad (22)$$

and know that all edges present in $G[Q_1, L_1]$ are coloured exclusively red and all edges present in $G[Q_2, L_1]$ are coloured exclusively blue. In this case, we disregard L_2 and consider $G[L_1]$.

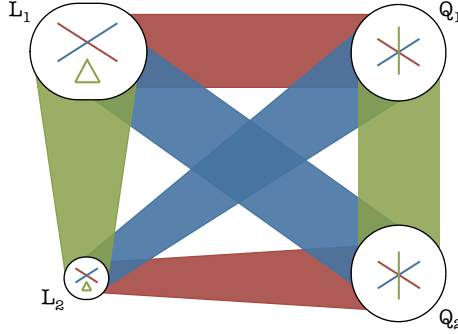


Figure 12: Colouring of $G[L \cup Q]$ in Case E.i.b.ii.

Since $\eta \leq (\alpha_1/1000)^2$, by (20) and (22),

$$|L_1| \geq |Q_1| + 200\eta^{1/2}k, \quad |L_1| \geq |Q_2| + 200\eta^{1/2}k. \quad (23)$$

Suppose there exists a red matching R_S on $196\eta^{1/2}k$ vertices in $G[L_1]$. Then, by (21) and (23),

$$|L_1 \setminus V(R_S)|, |Q_1| \geq (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k.$$

Thus, by Lemma 2.8, there exists a red connected-matching R_L on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices in $G[L_2, Q_1]$ which shares no vertices with R_S . Since all edges present in $G[L_1, Q_1]$ are coloured exclusively red and G is $4\eta^4k$ -almost-complete, R_S and R_L belong to the same red component and, thus, together form a red connected-matching on at least α_1k vertices. Therefore, the largest red matching in $G[L_1]$ spans at most $196\eta^{1/2}k$ vertices. Similarly, the largest blue connected-matching in $G[L_1]$ spans at most $196\eta^{1/2}k$ vertices. Thus, after discarding at most $392\eta^{1/2}k$ vertices from L_1 , we may assume that all edges in $G[L]$ are coloured green. Thus, we have obtained, as a subgraph of G , a graph in

$$\mathcal{K}\left(\left(\frac{1}{2}\alpha_1 - 97\eta^{1/2}\right)k, \left(\frac{1}{2}\alpha_2 - 97\eta^{1/2}\right)k, (\alpha_3 - 502\eta^{1/2})k, 4\eta^4k\right).$$

Case E.i.b.iii: $|L_1| \geq |Q_i| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$.

Suppose that $|L_1| \geq |Q_1| \geq |L_2| \geq |Q_2|$. Then, by (19a)–(19d) and (E4b'), we obtain

$$\begin{aligned} \left(\frac{1}{2}\alpha_1 - 97\eta^{1/2}\right)k &\leq |Q_1| \leq \left(\frac{1}{2}\alpha_1 + \eta^{1/2}\right)k, \\ \left(\frac{1}{2}\alpha_2 - 97\eta^{1/2}\right)k &\leq |Q_2| \leq \left(\frac{1}{2}\alpha_2 + \eta^{1/2}\right)k. \end{aligned}$$

Then, given the sizes of M_{11} , M_{12} , M_{21} and M_{22} , there can be no red or blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$ and we may assume that $\alpha_1 \leq \alpha_2 + 196\eta^{1/2}k$. Thus, we have found, as a subgraph of G , a graph in

$$\mathcal{K}^*((\frac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - \frac{1}{2}\alpha_1 - 12\eta^{1/2})k, (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4k) \subset \mathcal{K}_2^*.$$

Suppose instead that $|L_1| \geq |Q_2| \geq |L_2| \geq |Q_1|$. Then, by (19a)–(19d) and (E4b'), we obtain

$$\begin{aligned} (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |Q_1| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |Q_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k. \end{aligned}$$

Again, there can be no red or blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$. Thus, we have found, as a subgraph of G , a graph in

$$\mathcal{K}^*((\frac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - \frac{1}{2}\alpha_2 - 12\eta^{1/2})k, (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4k) \subset \mathcal{K}_2^*.$$

Case E.i.b.iv: $|Q_i| \geq |L_1| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$.

Suppose $|Q_1| \geq |L_1| \geq |L_2| \geq |Q_2|$. Then, $|Q_1| \geq \frac{1}{2}|L_1| + \frac{1}{2}|L_2| = \frac{1}{2}|L|$ and, since $\eta \leq (\alpha_2/100)^2$, by (E4a'), we have

$$|Q_1| \geq (\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 - 5\eta^{1/2})k \geq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 5\eta^{1/2})k \geq (\frac{1}{2}\alpha_1 + \eta^{1/2})k.$$

Also, by (E4a'), we have

$$|L| \geq (\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2})k \geq (\alpha_1 + \alpha_2 - 10\eta^{1/2})k.$$

Thus, either

$$|L_1| \geq (\alpha_1 - 5\eta^{1/2})k, \quad \text{or} \quad |L_2| \geq (\alpha_2 - 5\eta^{1/2})k.$$

Recall that all edges present in $G[L_1, Q_1]$ are coloured exclusively red and all edges present in $G[L_2, Q_1]$ are coloured exclusively blue. Thus, by Lemma 2.8, there exists either a red connected-matching on at least $\alpha_1 k$ vertices in $G[L_1, Q_1]$ or a blue connected-matching on at least $\alpha_2 k$ vertices in $G[L_1, Q_1]$.

The result is the same in the case that $|Q_2| \geq |L_1| \geq |L_2| \geq |Q_1|$.

Case E.i.c: $G[L, Q]$ contains red and blue ‘stars’ centred in L .

We continue to assume that F , the largest green connected-matching in G , spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and is contained in an odd component of G and that we have a partition of $V(G)$ into $L \cup P \cup Q$, satisfying (E4a') and (E4b') such that all edges present in $G[L, Q]$ are coloured red or blue. Additionally, in this case, we have vertices $v_r, v_b \in L$ such that v_r has red edges to all

but $36\eta^{1/2}k$ vertices in Q and v_b has blue edges to all but $36\eta^{1/2}k$ vertices in Q . Observe that the existence of v_r means that

(E5') $G[Q]$ has a red effective-component on at least $|Q| - 36\eta^{1/2}k$ vertices.

The proof then proceeds as in Case E.i.a but with (E5) replaced by (E5'). The result is the same.

Case E.i.d: $G[L, Q]$ contains red and blue ‘stars’ centred in Q .

We continue to assume that F , the largest green connected-matching in G , spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and is contained in an odd component of G and that we have a partition of $V(G)$ into $L \cup P \cup Q$, satisfying (E4a') and (E4b') such that all edges present in $G[L, Q]$ are coloured red or blue. Additionally, in this case, we have vertices $v_r, v_b \in Q$ such that v_r has red edges to all but $36\eta^{1/2}k$ vertices in L and v_b has blue edges to all but $36\eta^{1/2}k$ vertices in L . Observe that the existence of v_r means that

(E5'') $G[L]$ has a red effective-component on at least $|L| - 36\eta^{1/2}k$ vertices.

The proof then proceeds as in Case E.i.a but with (E5) replaced by (E5''). The result is the same.

Case E.ii: $|Q| \leq 95\eta^{1/2}k$.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ such that:

- (E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;
- (E2) every vertex in P has a green edge to M ;
- (E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$.

Recall also that

$$(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3k, \quad (\text{E4a})$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b})$$

Here, we consider the case when Q is sufficiently small to be disregarded. In that case, provided $|Q| \leq 95\eta^{1/2}k$, from (E4b), we have

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 96\eta^{1/2})k \leq |P| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b''})$$

By (E3), every edge in $G[P]$ is red or blue. Then, since G is $4\eta^4k$ -almost-complete, by Lemma 2.11, the largest monochromatic component F_P in $G[P]$ contains at least $|P| - \eta^{1/2}k$ vertices. Suppose that that this component is red.

We consider the largest red matching R in $G[N, P]$ and partition N into $N_1 \cup N_2$ and P into $P_1 \cup P_2$, where $N_1 = N \cap V(R)$, $N_2 = N \setminus N_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \setminus Q_1$. Then, by maximality of R , all edges present in $G[N_2, P_2]$ are coloured exclusively blue.

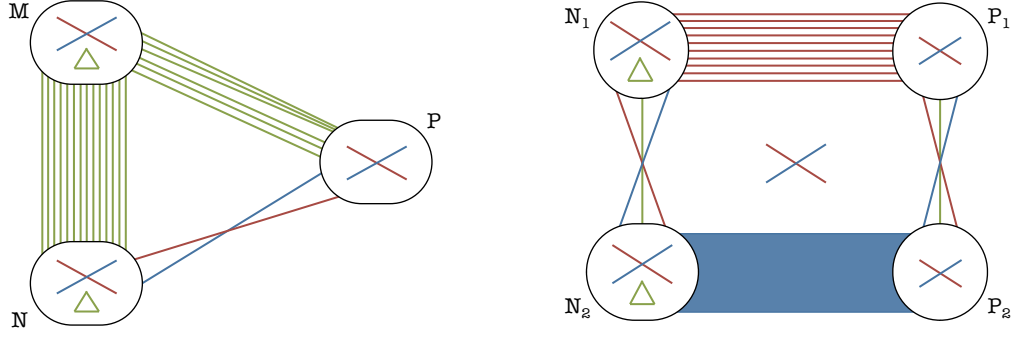


Figure 13: Decompositions in Case E.ii.

Since P has a large red connected-component, all but $\eta^{1/2}k$ of the edges of R belong to the same red-component and thus form a red connected-matching. Thus, in order to avoid having a red connected-matching on at least $\alpha_1 k$ vertices, we have $|P_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k$.

Suppose that $|N_2| \geq |P_2|$. Then, we have $|P_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$, since otherwise, by Lemma 2.8, $G[N_2, P_2]$ would contain a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, by (E4a) and (E4b''), we have

$$\left. \begin{aligned} (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |P_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |P_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |N_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |N_2|. \end{aligned} \right\} \quad (24)$$

If instead $|N_2| \leq |P_2|$, then, by Lemma 2.8, we have $|N_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$ since, otherwise, by Lemma 2.8, $G[N_2, P_2]$ contains a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, by (E4a) and (E4b''), we have

$$\left. \begin{aligned} (\frac{3}{4}\alpha_1 - \frac{1}{4}\alpha_2 - 6\eta^{1/2})k &\leq |P_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |P_2| \leq (\frac{3}{4}\alpha_1 - \frac{1}{4}\alpha_2 + 11\eta^{1/2})k, \\ (\frac{3}{4}\alpha_1 - \frac{1}{4}\alpha_2 - 6\eta^{1/2})k &\leq |N_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 - 6\eta^{1/2})k &\leq |N_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k, \end{aligned} \right\} \quad (25)$$

which yields a contradiction unless $\alpha_1 \leq \alpha_2 + 28\eta^{1/2}$.

Six of the eight bounds obtained in (25) are stronger than the corresponding bounds obtained in (24). The seventh is weaker but can be written in a similar form. Thus, we will combine the two cases and continue under the assumption that

$$\left. \begin{aligned} (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |P_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |P_2| \leq (\frac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |N_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |N_2|. \end{aligned} \right\} \quad (26)$$

Suppose there exists a blue matching B_1 on $198\eta^{1/2}k$ vertices in $G[N_2, P_1]$ and a blue matching B_2 on $198\eta^{1/2}k$ vertices in $G[N_1, P_2]$. Then, defining $\tilde{N} = N_2 \setminus V(B_1)$ and $\tilde{P} = P_2 \setminus V(B_2)$, we have $|\tilde{N}|, |\tilde{P}| \geq (\frac{1}{2}\alpha_2 - 197\eta^{1/2})k$ and, thus, by Lemma 2.8, there exists a blue connected-matching B_3 on at least $(\alpha_2 - 396\eta^{1/2})k$ vertices in $G[\tilde{N}, \tilde{P}]$. Since $G_2[N_2, P_2]$ is $4\eta^4k$ -almost-complete, all vertices in $N_2 \cup P_2$ belong to the same blue component. Thus, $B_1 \cup B_2 \cup B_3$ forms a blue connected-matching on at least α_2k vertices.

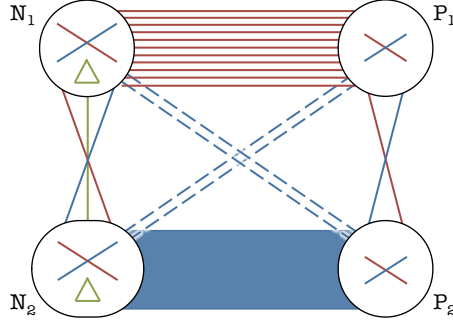


Figure 14: Colouring of the edges of $G[L_1, Q_2], G[L_2, Q_1]$.

Therefore, we proceed considering the following two subcases:

- (a) the largest blue matching in $G[N_2, P_1]$ spans at most $198\eta^{1/2}k$ vertices;
- (b) the largest blue matching in $G[N_1, P_2]$ spans at most $198\eta^{1/2}k$ vertices.

Case E.ii.a: Most edges in $G_2[N_2, P_1]$ are red.

Since the largest blue matching in $G[N_2, P_1]$ spans at most $198\eta^{1/2}k$ vertices, we can discard at most $99\eta^{1/2}k$ vertices from each of N_2 and P_1 so that all edges present in $G[N_2, P_1]$ are coloured exclusively red. In order to retain the equality $|N_1| = |P_1|$ and the property that every vertex in N_1 belongs to an edge of R , we discard from N_1 each vertex whose R -mate in P_1 has already been discarded. Recalling (26), we then have

$$\begin{aligned} (\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\leq |N_2|. \end{aligned}$$

Now, suppose there exists a red matching R_U on $396\eta^{1/2}k$ vertices in $G[N_1, P_2]$. Observe that there exists a set R^- of $198\eta^{1/2}k$ edges belonging to R such that $N_1 \cap V(R_S) = N_1 \cap V(R^-)$. Then, we have $|N_2|, |P_1 \cap V(R^-)| \geq 198\eta^{1/2}k$ so, by Lemma 2.8, there exists a red connected-matching R_V on at least $394\eta^{1/2}k$ vertices in $G[N_2, P_1 \cap V(R^-)]$. Then, since all edges present in $G[N_2, P_1]$ are coloured exclusively red and every vertex in N_1 has a red neighbour in P_1 , $(R \setminus R^-)$, R_U and R_V belong to the same red-component and, thus, together form a red-connected-matching on at least $2(\frac{1}{2}\alpha_1 - 196\eta^{1/2}k - 199\eta^{1/2}k) + 396\eta^{1/2}k + 394\eta^{1/2}k \geq \alpha_1k$ vertices. Thus, we can discard at most

$198\eta^{1/2}k$ vertices from each of N_1 and P_2 so that all edges present in $G[N_1, P_2]$ are coloured exclusively blue. We then have

$$\begin{aligned} (\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_1 - 394\eta^{1/2})k &\leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\leq |N_2|. \end{aligned}$$

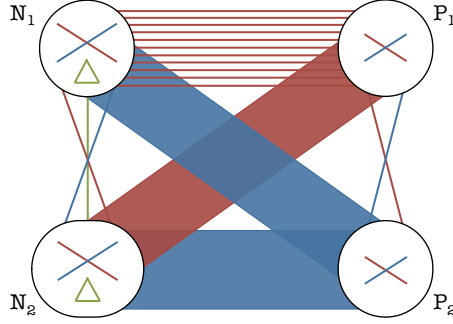


Figure 15: Colouring of the edges of $G[N_1, P_2]$.

Observe then that, since $|N_2|, |P_2| \geq (\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k$, by Lemma 2.8, there exists a blue connected-matching on at least $(\alpha_2 - 592\eta^{1/2})k$ vertices in $G[N_2, P_2]$. Thus, since all edges present in $G[N, P_2]$ are coloured exclusively blue, if there existed a blue matching on $592\eta^{1/2}k$ vertices in $G[N_1, P_1]$, we would have a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, after discarding at most $296\eta^{1/2}k$ vertices from each of P_1 and N_1 , we may assume that all edges in $G[P_1, N_1]$ are coloured exclusively red and that

$$\left. \begin{aligned} (\tfrac{1}{2}\alpha_1 - 492\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_1 - 690\eta^{1/2})k &\leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\leq |N_2|. \end{aligned} \right\} \quad (27)$$

Then, since $\eta \leq (\alpha_2/20000)^2$, by (27), we have

$$|N| = |N_1| + |N_2| \geq |P_1| + 5000\eta^{1/2}k, \quad (28)$$

$$|N| = |N_1| + |N_2| \geq |P_2| + 5000\eta^{1/2}k. \quad (29)$$

In particular, $|N| \geq |P_2|$ so, by Lemma 2.8, we have

$$|P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k. \quad (30)$$

Having determined the colouring of the red-blue graph $G[N, P]$, we now expand our sights to $G[M \cup N]$ and $G[M, P]$, each of which can include green edges:

Suppose there exists a red matching R_A on $986\eta^{1/2}k$ vertices in $G[N] \cup G[M, N]$. Then, by (27)

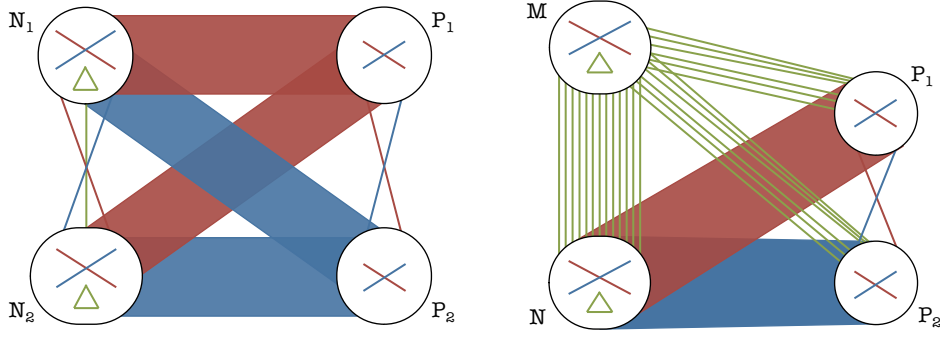


Figure 16: Colouring after three rounds of discarding vertices.

and (28), we have $|N \setminus V(R_A)| \geq |P_1| \geq (\frac{1}{2}\alpha_1 - 492\eta^{1/2})k$ so, by Lemma 2.8, there exists a red connected-matching R_B on at least $(\alpha_1 - 986\eta^{1/2})k$ in $G[N \setminus V(R_A), P_1]$. Since all edges in $G[N, P_1]$ are coloured exclusively red, R_A and R_B belong to the same red component and, thus, together form a red connected-matching on at least $\alpha_1 k$ vertices. Similarly, if there exists a blue matching on at least $594\eta^{1/2}k$ vertices in $G[N] \cup G[M, N]$, then this can be used along with $G[N, P_2]$ to give a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $790\eta^{1/2}k$ vertices from M and at most $2370\eta^{1/2}k$ vertices from N , we may assume that all edges present in $G[N]$ and $G[M \cup N]$ are green.

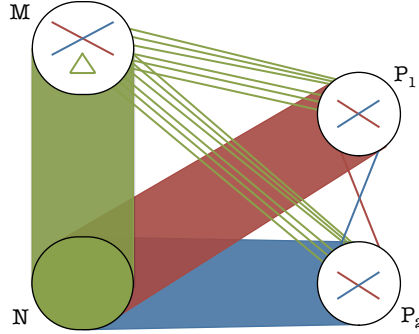


Figure 17: Colouring of the edges of $G[N] \cup G[N, M]$.

After discarding these vertices, we have

$$\left. \begin{aligned} (\frac{1}{2}\alpha_3 - 800\eta^{1/2})k &\leq |M| \leq \frac{1}{2}\alpha_3 k, \\ (\frac{1}{2}\alpha_3 - 3300\eta^{1/2})k &\leq |N| \leq \frac{1}{2}\alpha_3 k. \end{aligned} \right\} \quad (31)$$

Next, suppose there exists a green matching G_S on $8240\eta^{1/2}k$ vertices in $G[M, P]$. By (31), we have $|M \setminus V(G_S)| \geq (\frac{1}{2}\alpha_3 - 4920\eta^{1/2})k$. Then, taking \tilde{N} to be any subset of $(\frac{1}{2}\alpha_3 - 4920\eta^{1/2})k$ vertices in N , by Lemma 2.8, there exists a green connected-matching G_L on at least $(\alpha_3 - 9842\eta^{1/2})k$ vertices in $G[M \setminus V(G_S), \tilde{N}]$. Also, since $|N \setminus \tilde{N}| \geq 1618\eta^{1/2}k$, by Theorem 2.1, there exists a green matching G_T on $1610\eta^{1/2}k$ vertices in $G[N \setminus \tilde{N}]$. Then, together G_S , G_L and G_T form a green connected-matching on at least $\alpha_3 k$ vertices which is odd by the definition of the decomposition. Thus, we may discard at most $4120\eta^{1/2}k$ vertices from each of M and P such that none of the edges present in $G[P, M \cup N]$ are coloured green.

Then, recalling (27), (30) and (31), we have

$$\left. \begin{aligned} (\tfrac{1}{2}\alpha_1 - 4612\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 4415\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_3 - 4920\eta^{1/2})k &\leq |M| \leq \tfrac{1}{2}\alpha_3 k, \\ (\tfrac{1}{2}\alpha_3 - 3300\eta^{1/2})k &\leq |N| \leq \tfrac{1}{2}\alpha_3 k. \end{aligned} \right\} \quad (32)$$

Now, suppose there exists a red matching R_S on $9230\eta^{1/2}k$ vertices in $G[M, P_2]$. By (28) and (32), we have $|N| \geq |P_1| \geq (\tfrac{1}{2}\alpha_1 - 4612\eta^{1/2})k$. So, by Lemma 2.8, there exists a red connected-matching R_L on at least $2|P_1| - 2\eta^{1/2}k \geq (\alpha_1 - 9226\eta^{1/2})k$ vertices in $G[N, P_1]$. Since $G[P]$ has a red effective-component on at least $|P| - \eta^{1/2}k$, R_L belongs to the same red component as at least $4614\eta^{1/2}k$ of the edges of R_S , thus giving a red connected-matching on at least $\alpha_1 k$ vertices in $G[M, P_2] \cup G[N, P_1]$. Therefore, after discarding at most $4615\eta k$ vertices from each of M and P_2 , we may assume that all edges present in $G[M, P_2]$ are coloured exclusively blue.

We then have

$$\left. \begin{aligned} (\tfrac{1}{2}\alpha_1 - 4612\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 9030\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_3 - 9335\eta^{1/2})k &\leq |M| \leq \tfrac{1}{2}\alpha_3 k, \\ (\tfrac{1}{2}\alpha_3 - 3300\eta^{1/2})k &\leq |N| \leq \tfrac{1}{2}\alpha_3 k. \end{aligned} \right\} \quad (33)$$

Similarly, suppose there exists a blue matching B_S on $18062\eta^{1/2}k$ vertices in $G[M, P_1]$. By (29) and (33), $|N| \geq |P_2| \geq (\tfrac{1}{2}\alpha_1 - 9030\eta^{1/2})k$. So, by Lemma 2.8, there exists a blue connected-matching B_L on at least $2|P_2| - 2\eta^{1/2}k \geq (\alpha_1 - 18062\eta^{1/2})k$ vertices in $G[N, P_2]$. Then, since G is $4\eta^4 k$ -almost-complete and all edges present in $G[M, P_2]$ are coloured exclusively blue, B_L and B_S belong to the same blue component of G , and thus, together, form a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $9031\eta k$ vertices from each of M and P_1 , we may assume that all edges present in $G[M, P_1]$ are coloured exclusively red. We then have

$$\left. \begin{aligned} (\tfrac{1}{2}\alpha_1 - 13643\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 9030\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_3 - 18366\eta^{1/2})k &\leq |M| \leq \tfrac{1}{2}\alpha_3 k, \\ (\tfrac{1}{2}\alpha_3 - 3300\eta^{1/2})k &\leq |N| \leq \tfrac{1}{2}\alpha_3 k. \end{aligned} \right\} \quad (34)$$

Finally, since $|N| \geq |P_1|, |P_2|$, if there existed a red matching on at least $27288\eta^{1/2}k$ vertices in $G[M]$ or a blue matching on at least $18062\eta^{1/2}k$ vertices in $G[M]$, then we could obtain a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $45350\eta^{1/2}k$ further vertices from M , we may assume that all edges present in $G[M \cup N]$ are green.

In summary, we now have

$$|P_1| \geq (\tfrac{1}{2}\alpha_1 - 13643\eta)k, \quad |P_2| \geq (\tfrac{1}{2}\alpha_2 - 9030\eta)k, \quad |M \cup N| \geq (\alpha_3 - 67216\eta^{1/2})k,$$

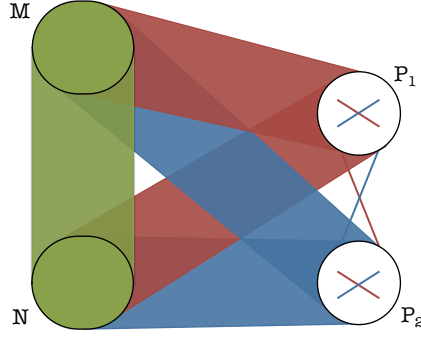


Figure 18: Final colouring in Case E.ii.a.

and know that all edges present in $G[M \cup N, P_1]$ are coloured exclusively red, all edges present $G[M \cup N, P_2]$ are coloured exclusively blue and all edges in $G[M \cup N]$ are coloured exclusively green.

We have thus found, as a subgraph of G , a graph belonging to

$$\mathcal{K} \left(\left(\frac{1}{2}\alpha_1 - 14000\eta^{1/2} \right)k, \left(\frac{1}{2}\alpha_2 - 14000\eta^{1/2} \right)k, (\alpha_3 - 68000\eta^{1/2})k, 4\eta^4k \right).$$

Case E.ii.b: Most edges in $G_2[N_1, P_2]$ are red.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ such that:

- (E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;
- (E2) every vertex in P has a green edge to M ;
- (E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$.

Recall also that

$$(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3k, \quad (\text{E4a})$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b})$$

Furthermore, recall that the largest red matching R in $G[N, P]$ defines a partition of N into $N_1 \cup N_2$ and P into $P_1 \cup P_2$ such that the edges of R belong $G[N_1, P_1]$, all edges present in $G[N_2, P_2]$ are coloured exclusively blue and that

$$\begin{aligned} (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |P_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |P_2| \leq (\frac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k &\leq |N_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\frac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |N_2|. \end{aligned}$$

Additionally, in this case, we assume that the largest blue matching in $G[N_1, P_2]$ spans at most $198\eta^{1/2}k$ vertices. Thus, we can discard at most $99\eta^{1/2}k$ vertices from each of N_1 and P_2 so that all edges present in $G[N_1, P_2]$ are coloured exclusively red.

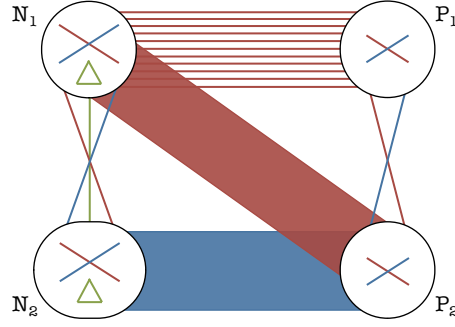


Figure 19: Initial colouring in Case E.ii.b.

In order to retain the equality $|N_1| = |P_1|$ and the property that every vertex in P_1 belongs to an edge of R , we discard from P_1 each vertex whose R -mate in N_1 has already been discarded. We then have

$$\begin{aligned} (\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k &\leq |N_2|. \end{aligned}$$

Then, suppose there exists a red matching on $396\eta^{1/2}k$ vertices in $G[N_2, P_1]$. Such a matching could be used together with $G[N_1, P_1]$ and $G[N_1, P_2]$ to give a red connected-matching on at least $\alpha_1 k$ vertices. Thus, we can discard at most $198\eta^{1/2}k$ vertices from each of N_2 and P_1 so that all edges present in $G[N_2, P_1]$ are coloured exclusively blue.

We then have

$$\begin{aligned} (\tfrac{1}{2}\alpha_1 - 394\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k &\leq |N_2|. \end{aligned}$$

Then, if there existed a blue matching on $592\eta^{1/2}k$ vertices in $G[P_1, N_1]$, this could be used along with $G[P, N_2]$ to give a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $296\eta^{1/2}k$ vertices from each of N_1 and P_1 , we may assume that all edges present in $G[N_1, P_1]$ are coloured exclusively red and that

$$\left. \begin{aligned} (\tfrac{1}{2}\alpha_1 - 690\eta^{1/2})k &\leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_1 - 492\eta^{1/2})k &\leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k &\leq |N_2|. \end{aligned} \right\} \quad (35)$$

Observe, now, that, given any subset P' of $(\tfrac{1}{2}\alpha_1 - 492\eta^{1/2})k$ vertices from P , by Lemma 2.8, there exists a red connected-matching on at least $(\alpha_1 - 986\eta^{1/2})k$ vertices in $G[N_1, P']$. Thus, the largest red

matching in $G[P]$ spans at most $986\eta^{1/2}k$ vertices. Similarly, given any subset P'' of $(\frac{1}{2}\alpha_1 - 295\eta^{1/2})k$ vertices from P , we can find a blue connected-matching on at least $(\alpha_1 - 592\eta^{1/2})k$ vertices in $G[N_2, P'']$. Thus, the largest blue matching in $G[P]$ spans at most $592\eta^{1/2}k$ vertices. Thus, since there are no green edges within $G[P]$, we have $|P| \leq 1600\eta^{1/2}k$, which, since $\eta \leq (\alpha_1/10000)^2$, contradicts (35), completing Case E.ii.b.

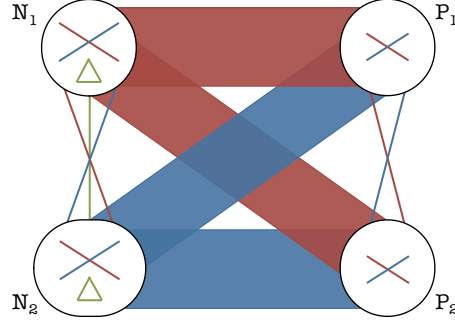


Figure 20: Colouring of the edges of $G[N, P]$.

At the beginning of Case E.ii, we assumed that F_P , the largest monochromatic component in $G[P]$, was red. If instead that monochromatic component is blue, then the proof proceeds exactly as above following the same steps with the roles of red and blue exchanged and with α_1 and α_2 exchanged. The result is identical.

Case E.iii: $|P|, |Q| \geq 95\eta^{1/2}k$.

We now consider the case when neither P nor Q is trivially small. This case is fairly involved, combining elements of Case E.i and Case E.ii with new arguments. However, because neither P nor Q is trivially small, we can exploit the structure of the two coloured graph $G[P \cup Q]$.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ such that:

- (E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;
- (E2) every vertex in P has a green edge to M ;
- (E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$.

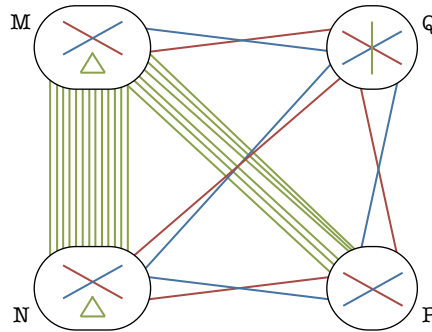


Figure 21: Decomposition in Case E.iii.

Recall also that

$$(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3 k, \quad (\text{E4a})$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b})$$

In this case, we assume that $|P|, |Q| \geq 95\eta^{1/2}k$. Recall that $G[P \cup Q]$ is $4\eta^4 k$ -almost-complete. Then, since $|P| + |Q| \geq 190\eta^{1/2}k$, we have $4\eta^4 k \leq \eta^2(|P| + |Q| - 1)$ and, thus, $G[P \cup Q]$ is $(1 - \eta^2)$ -complete. Since $\alpha_2 \leq \alpha_1 \leq 1$, recalling (E4b), we have $|P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k \leq (1 + 5\eta^{1/2})k$ and so

$$|P|, |Q| \geq 95\eta^{1/2}k \geq 4\eta^{1/2}(1 + 5\eta^{1/2}) \geq 4\eta^{1/2}(|P| + |Q|) \geq 4\eta(|P| + |Q|).$$

Thus, provided $k \geq 1/(190\eta^{5/2})$, we may apply Lemma 2.12 to $G[P \cup Q]$, giving rise to two possibilities:

- (a) $P \cup Q$ contains a monochromatic component F on at least $|P \cup Q| - 8\eta k$ vertices;
- (b) there exist vertices $w_r, w_b \in Q$ such that w_r has red edges to all but $8\eta k$ vertices in P and w_b has blue edges to all but $8\eta k$ vertices in P .

Case E.iii.a: $P \cup Q$ has a large monochromatic component.

Suppose that F , the largest monochromatic component in $P \cup Q$ is red. We consider $G[M, Q]$ and $G[N, P]$, both of which have only red and blue edges, and let R be the largest red matching in $G[M, Q] \cup G[N, P]$. We partition each of M, N, P, Q into two parts such that $M_1 = M \cap V(R)$, $M_2 = M \setminus M_1$, $N_1 = N \cap V(R)$, $N_2 = N \setminus N_1$, $P_1 = P \cap V(R)$, $P_2 = P \setminus P_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \setminus Q_1$. Observe that, by maximality of R , all edges present in $G[P_2, N_2]$ and $G[Q_2, N_2]$ are blue.

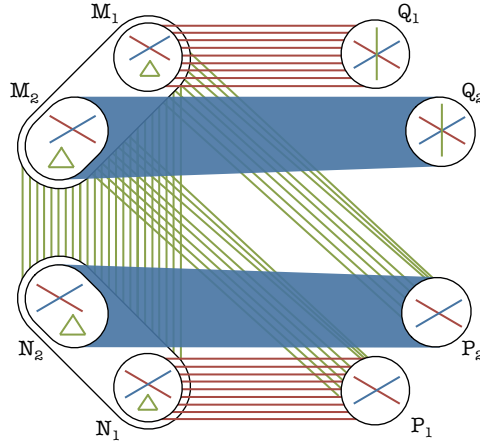


Figure 22: Decomposition into eight parts.

Notice that, since G is $4\eta^4 k$ -almost-complete, $G[M_2 \cup Q_2]$ and $G[N_2 \cup P_2]$ each have a single blue component. We then consider two subcases:

- (i) P_2 and Q_2 belong to the same blue component of G ;
- (ii) P_2 and Q_2 belong to different blue components of G .

Case E.iii.a.i: P_2 and Q_2 belong to the same blue component.

Since F , the largest red component in $P \cup Q$, contains at least $|P \cup Q| - 8\eta k$ vertices, all but at most $8\eta k$ of the edges of R belong to F . Thus, since $|M_1| = |Q_1|$ and $|N_1| = |P_1|$, we have a red connected-matching on at least $2(|P_1| + |Q_1| - 8\eta k)$ vertices and so we may assume that

$$|P_1| + |Q_1| \leq (\tfrac{1}{2}\alpha_1 + 8\eta)k \quad (36)$$

in order to avoid having a red connected-matching on at least $\alpha_1 k$ vertices.

Recalling (E4a) and (E4b), since $\eta \leq (\alpha_1/100)^2$, we have

$$|M| + |N| \geq |P| + |Q| + 80\eta^{1/2}k$$

and also

$$|M|, |N| \geq \tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 5\eta^{1/2}k \geq \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 5\eta^{1/2}k \geq |P| + |Q| - 10\eta^{1/2}.$$

Thus, since $|M_1| = |Q_1|$, $|N_1| = |P_1|$ and $|P|, |Q| \geq 95\eta^{1/2}k$, we have

$$|M_2| \geq |P| + |Q_2| - 10\eta^{1/2}k \geq |Q_2| + 85\eta^{1/2}k, \quad (37a)$$

$$|N_2| \geq |P_2| + |Q| - 10\eta^{1/2}k \geq |P_2| + 85\eta^{1/2}k. \quad (37b)$$

In particular, we have $|M_2| \geq |Q_2|$ and $|N_2| \geq |P_2|$. Thus, since P_2 and Q_2 belong to the same effective-blue component, by Lemma 2.8, there exists a blue connected-matching on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices in $G[N_2, P_2] \cup G[M_2, Q_2]$. Thus, we may assume that

$$|P_2| + |Q_2| \leq (\tfrac{1}{2}\alpha_2 + 2\eta)k, \quad (38)$$

in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices.

Then, by (E4a), (E4b), (36) and (38), we have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 3\eta)k, \quad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 9\eta)k. \quad (39)$$

We now proceed to determine the coloured structure of G . We begin by proving the following claim whose proof follows the same three steps as that of Claim 3.1. Considering in parallel $G[M, Q]$ and $G[N, P]$, the first step in the proof is to show that, after possibly discarding some vertices, all edges contained in $G[M_2, Q_1] \cup G[N_2, P_1]$ are coloured exclusively red, the second is to show that, after possibly discarding further vertices, all edges contained in $G[M_1, Q_2] \cup G[N_1, P_2]$ are coloured exclusively blue and the third is to show that, after possibly discarding still more vertices, all edges contained in $G[M_1, Q_1] \cup G[N_1, P_1]$ are coloured exclusively red.

Claim 3.2. *We may discard at most $70\eta k$ vertices from $P_1 \cup Q_1$, at most $48\eta^{1/2}k$ vertices from $P_2 \cup Q_2$, at most $94\eta^{1/2}k$ vertices from each of M_1 and N_1 and at most $11\eta k$ vertices from $M_2 \cup N_2$ such that, in what remains, all edges present in $G[M, Q_1] \cup G[N, P_1]$ are coloured exclusively red and all edges present in $G[M, Q_2] \cup G[N, P_2]$ are coloured exclusively blue.*

Proof. Suppose there exists a blue matching B_S on at least $22\eta k$ vertices in $G[M_2, Q_1] \cup G[N_2, P_1]$. By (37a) and (37b), we have $|M_2 \setminus V(B_S)| \geq |Q_2|$ and $|N_2 \setminus V(B_S)| \geq |P_2|$. Thus, since P_2 and Q_2 belong to the same blue component of G , applying Lemma 2.8 to each of $G[M_2 \setminus V(B_S), Q_2]$ and $G[N_2 \setminus V(B_S), P_2]$ gives a blue connected-matching B_L in $G[M_2 \setminus V(B_S), Q_2] \cup G[N_2 \setminus V(B_S), P_2]$ on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices which belongs to the same blue component of G as B_S but shares no vertices with it. Thus, $B_L \cup B_S$ forms a blue connected-matching on at least $2|P_2| + 2|Q_2| + 18\eta k \geq \alpha_2 k$ vertices. Therefore, after discarding at most $11\eta k$ vertices from each of $M_2 \cup N_2$ and $P_1 \cup Q_1$, we may assume that all edges present in $G[M_2, Q_1] \cup G[N_2, P_1]$ are coloured exclusively red.

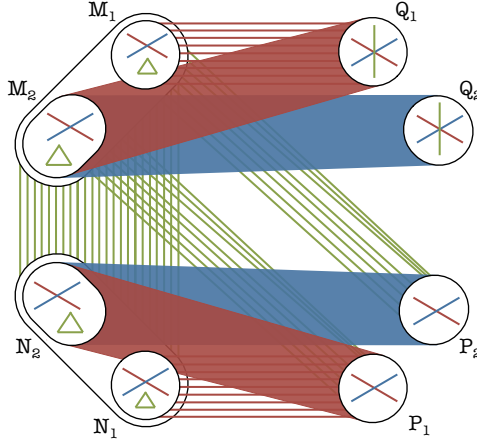


Figure 23: Colouring of $G[M_2, Q_1] \cup G[N_2, P_1]$.

Recalling (39), we then have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 14\eta)k, \quad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 9\eta)k.$$

In order to retain the equalities $|M_1| = |Q_1|$ and $|N_1| = |P_1|$ and the property that every vertex in $M_1 \cup N_1$ belongs to an edge of R , we discard from $M_1 \cup N_1$ each vertex whose R -mate in $P_1 \cup Q_1$ has already been discarded. Thus, we discard at most a further $11\eta k$ vertices.

Then, if either $G[N_1, P_2]$ or $G[M_1, Q_2]$ contains a red matching on at least $48\eta k$ vertices in $G[N_1, P_2]$, then we may construct a red connected-matching on at least $\alpha_1 k$ vertices as follows:

Suppose that there exists a red matching R_P on $48\eta k$ vertices in $G[N_1, P_2]$. Then, observe that there exists a set R^- of $24\eta k$ edges belonging to R such that $N_1 \cap V(R_P) = N_1 \cap V(R^-)$. Define $\tilde{P} = P_1 \cap V(R^-)$ and consider $G[N_2, \tilde{P}]$. Since $|N_2|, |\tilde{P}| \geq 24\eta k$ and $G[N_2, \tilde{P}]$ is $4\eta^4 k$ -almost-complete, we may apply Lemma 2.8 to find a red connected-matching R_S on at least $46\eta k$ vertices in $G[N_2, \tilde{P}]$. Then, recalling that $P \cup Q$ has a red effective-component including all but $8\eta k$ of its vertices, we have a red connected-matching $R^* \subseteq (R \setminus R^-) \cup R_S \cup R_P$ on at least

$$\begin{aligned} 2(|P_1 \setminus V(R^-)| + |Q_1| + |P_1 \cap V(R_S)| + |P_2 \cap V(R_P)| - 8\eta k) \\ \geq 2(\tfrac{1}{2}\alpha_1 - 39\eta + 23\eta + 24\eta - 8\eta)k \geq \alpha_1 k \text{ vertices} \end{aligned}$$

in $G[N_1 \setminus V(R_P), P_1 \setminus V(R_S)] \cup G[M_1, Q_1] \cup G[N_2, P_1 \cap V(R_S)] \cup G[N_1 \cap V(R_P), P_2]$.

Similarly, suppose that there exists a red matching R_Q on $48\eta k$ vertices in $G[M_1, Q_2]$. Then, observe that there exists a set $R^=$ of $24\eta k$ edges belonging to R such that $M_1 \cap V(R_Q) = M_1 \cap V(R^=)$. Define $\tilde{Q} = Q_1 \cap V(R^=)$ and consider $G[M_2, \tilde{Q}]$. Since $|M_2|, |\tilde{Q}| \geq 24\eta k$ and $G[M_2, \tilde{Q}]$ is $4\eta^4 k$ -almost-complete, we may apply Lemma 2.8 to find a red connected-matching R_T on at least $46\eta k$ vertices in $G[M_2, \tilde{Q}]$. Then, recalling that $P \cup Q$ has a red effective-component including all but $8\eta k$ of its vertices, we have a red connected-matching $R^* \subseteq (R \setminus R^=) \cup R_T \cup R_Q$ on at least

$$2(|P_1| + |Q_1 \setminus V(R^=)| + |Q_1 \cap V(R_S)| + |Q_2 \cap V(R_Q)| - 8\eta k) \geq \alpha_1 k \text{ vertices}$$

in $G[N_1, P_1] \cup G[M_1 \setminus V(R_Q), Q_1 \setminus V(R_T)] \cup G[M_2, Q_1 \cap V(R_T)] \cup G[M_1 \cap V(R_Q), Q_2]$.

Thus, after discarding at most $24\eta k$ vertices from each of M_1, N_1, P_2 and Q_2 , we may assume that all edges present in $G[M_1, Q_2] \cup G[N_1, P_2]$ are coloured exclusively blue. We then have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 14\eta)k, \quad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 57\eta)k.$$

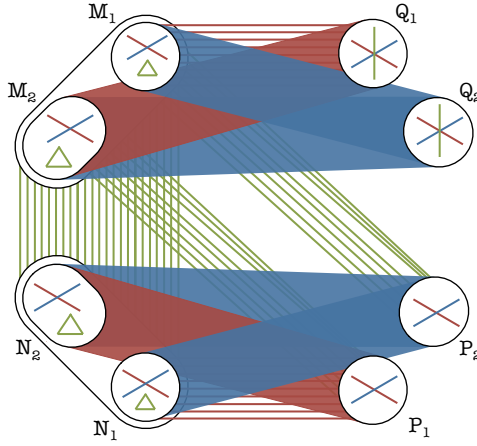


Figure 24: Colouring of $G[M_1, Q_2] \cup G[N_1, P_2]$.

Recalling (37a) and (37a), given that, so far, we have discarded at most $11\eta k$ vertices from $M_2 \cup N_2$, we have

$$|M_2| \geq |Q_2| + 14\eta^{1/2}k, \quad |N_2| \geq |P_2| + 14\eta^{1/2}k.$$

Thus, by Lemma 2.8, there exist blue connected-matchings B_1 spanning at least $2|Q_2| - 2\eta k$ vertices in $G[M_2, Q_2]$, and B_2 spanning at least $|P_2| - 2\eta k$ vertices in $G[N_2, Q_2]$. Recall that we assume that P_2 and Q_2 belong to the same blue effective-component. Then, since all edges present in $G[M, Q_2]$ and $G[N, P_2]$ are coloured blue, all vertices in $M \cup N \cup P_2 \cup Q_2$ belong to the same blue component of G and $B_1 \cup B_2$ forms a connected-matching on at least $2(|P_2| + |Q_2|) - 4\eta k \geq (\alpha_2 - 118\eta)k$ vertices in that component.

Thus, the largest blue matching in $G[M_1, Q_1] \cup G[N_1, P_1]$ spans at most than $118\eta k$ vertices. Therefore, after discarding at most $59\eta k$ vertices from each of $M_1 \cup N_1$ and $P_1 \cup Q_1$, we may assume that all edges present in $G[M_1, Q_1] \cup G[N_1, P_1]$ are coloured exclusively red, completing the proof of Claim 3.2. \square

In summary, we now know that all edges in $G[M, Q_1] \cup G[N, P_1]$ are coloured exclusively red and that all edges in $G[M, Q_2] \cup G[N, P_2]$ are coloured exclusively blue.

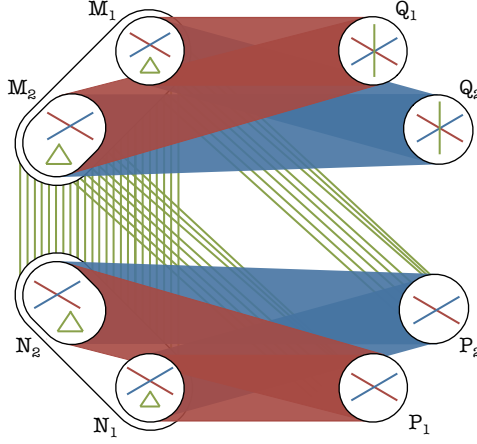


Figure 25: Colouring of $G[M, Q] \cup G[N, P]$.

Additionally, we have

$$\left. \begin{aligned} |M_1| + |M_2| &\geq |Q_1| + |Q_2| + 13\eta^{1/2}k, & |P_1| + |Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 73\eta)k, \\ |N_1| + |N_2| &\geq |P_1| + |P_2| + 13\eta^{1/2}k, & |P_2| + |Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 57\eta)k. \end{aligned} \right\} \quad (40)$$

We now move on to consider $G[M, P] \cup G[N, Q]$, taking the same approach as we did for $G[M, Q] \cup G[N, P]$ but recalling the possibility of green edges in $G[M, P]$. We prove the following claim:

Claim 3.3. *We may discard at most $145\eta k$ vertices from $P_1 \cup Q_1$, at most $84\eta^{1/2}k$ vertices from $P_2 \cup Q_2$ and at most $229\eta k$ vertices from $M \cup N$ such that, in what remains, there are no red edges present in $G[M, P_2] \cup G[N, Q_2]$ and no blue edges present in $G[M, P_1] \cup G[N, Q_1]$.*

Proof. Suppose that there exists a red matching R^\times on at least $168\eta k$ vertices in $G[M, P_2] \cup G[N, Q_2]$. Then, by (40), we have $|M \setminus V(R^\times)| \geq |Q_1|$ and $|N \setminus V(R^\times)| \geq |P_1|$. Thus, since all but at most $8\eta k$ vertices of $P \cup Q$ belong to the same red component of G , R^\times can be used along with edges from $G[N \setminus V(R^\times), P_1]$ and $G[M \setminus V(R^\times), Q_1]$ to form a red connected-matching on

$$(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k) + (168\eta k - 16\eta k) \geq \alpha_1 k$$

vertices. Thus, after discarding at most $84\eta k$ vertices from each of $M \cup N$ and $P_2 \cup Q_2$, we may assume that there are no red edges present in $G[M, P_2] \cup G[N, Q_2]$. In particular, since there are no green edges present in $G[N, Q]$, we know that all edges present in $G[N, Q_2]$ are coloured exclusively blue. We then have

$$\left. \begin{aligned} |M_1| + |M_2| &\geq |Q_1| + |Q_2| + 12\eta^{1/2}k, & |P_1| + |Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 73\eta)k, \\ |N_1| + |N_2| &\geq |P_1| + |P_2| + 12\eta^{1/2}k, & |P_2| + |Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 142\eta)k. \end{aligned} \right\} \quad (41)$$

Next, suppose that there exists a blue matching B^\times on at least $290\eta k$ vertices in $G[M, P_1] \cup G[N, Q_1]$. Then, by (41), we have $|M \setminus V(B^\times)| \geq |Q_2|$ and $|N \setminus V(B^\times)| \geq |P_2|$. Thus, since $M_2 \cup N_2 \cup P_2 \cup Q_2$ belong to the same component of G , B^\times can be used along with edges from $G[N \setminus V(B^\times), P_2]$ and

$G[M \setminus V(B^\times), Q_2]$ to give a blue connected-matching on at least

$$(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k) + 290\eta k \geq \alpha_2 k$$

vertices. Thus, after discarding at most $145\eta k$ vertices from each of $P_1 \cup Q_1$ and $N \cup M$, we may assume that there are no blue edges present in $G[M, P_1] \cup G[N, Q_1]$. In particular, since there are no green edges present in $G[N, Q]$, we know that all edges present in $G[N, Q_1]$ are coloured exclusively red.

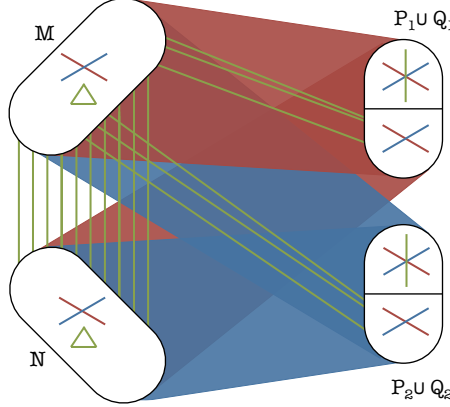


Figure 26: Colouring of $G[M, P] \cup G[N, Q]$ after Claim 3.3.

In summary, we have discarded at most $145\eta k$ vertices from $P_1 \cup Q_1$, at most $84\eta k$ vertices from $P_2 \cup Q_2$ and at most $229\eta k$ vertices from $M \cup N$. Having done so, we now know that there are no red edges present in $G[M, P_2] \cup G[N, Q_2]$ and that there are no blue edges present in $G[M, P_1] \cup G[N, Q_1]$. In particular, since we already knew that there are no green edges present in $G[N, Q]$, we know that all edges present in $G[N, Q_1]$ are coloured exclusively red and that all edges present in $G[N, Q_2]$ are coloured exclusively blue, thus completing the proof of Claim 3.3. \square

We now have

$$\left. \begin{aligned} |M| &\geq |Q_1| + |Q_2| + 11\eta^{1/2}k, & |P_1| + |Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 218\eta)k, \\ |N| &\geq |P_1| + |P_2| + 12\eta^{1/2}k, & |P_2| + |Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 142\eta)k. \end{aligned} \right\} \quad (42)$$

Finally, we turn our attention to $G[M \cup N]$, proving the following claim.

Claim 3.4. *We may discard at most $732\eta k$ vertices from $M \cup N$, such that, in what remains, all edges present in $G[M \cup N]$ are coloured exclusively green.*

Proof. Suppose that there exists a red matching R^\dagger on $442\eta k$ vertices in $G[M \cup N]$. Then, by (42), we have $|M \setminus V(R^\dagger)| \geq |Q_1|$, $|N \setminus V(R^\dagger)| \geq |P_1|$. Also, since all edges present in $G[M \cup N, Q_1]$ are coloured exclusively red, $M \cup N$ belongs to a single red component of G . Thus, by Lemma 2.8, there exists a red connected-matching R^\ddagger in $G[N \setminus V(R^\dagger), P_1] \cup G[M \setminus V(R^\dagger), Q_1]$ on at least $(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k)$ vertices. Then, since R^\dagger and R^\ddagger belong to the same red component but share no vertices, together they form a red connected-matching on at least $(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k) + 442\eta k \geq \alpha_1 k$ vertices.

Similarly, suppose that there exists a blue matching B^\dagger on $290\eta k$ vertices in $G[M \cup N]$. Then, by (42), we have $|M \setminus V(B^\dagger)| \geq |Q_2|$, $|N \setminus V(B^\dagger)| \geq |P_2|$. Since all edges present in $G[M \cup N, Q_2]$ are coloured exclusively blue, $M \cup N$ belongs to a single blue component of G . Then, by Lemma 2.8, there exists a blue connected-matching B^\ddagger in $G[N \setminus V(B^\dagger), P_2] \cup G[M \setminus V(B^\dagger), Q_2]$ on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices. Since B^\dagger and B^\ddagger belong to the same blue component but share no vertices, together they form a blue connected-matching on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k) + 290\eta k \geq \alpha_1 k$ vertices.

Thus, after discarding at most $732\eta k$ vertices from $M \cup N$, we can assume that all edges present in $G[M \cup N]$ are coloured exclusively green, completing the proof of Claim 3.4. \square

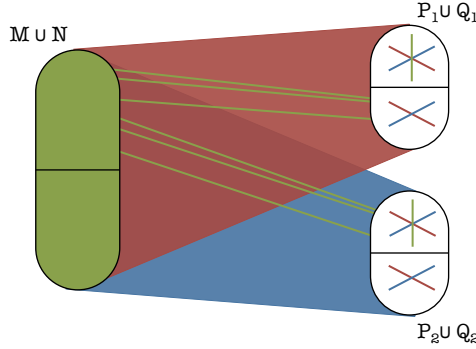


Figure 27: Colouring of $G[M \cup N]$.

Thus far, we have discarded at most $1200\eta k$ vertices from $M \cup N$. Recalling (E4), we now have $|M \cup N| \geq (\alpha_3 - 9\eta^{1/2})k$. Suppose there exists a green matching G^\dagger on $20\eta^{1/2}k$ vertices in $G[M \cup N, P \cup Q]$. Then, we have $|(M \cup N) \setminus V(G^\dagger)| \geq (\alpha_3 - 19\eta^{1/2})k$. By Theorem 2.1, since G is $4\eta^4 k$ -almost-complete, $G[(M \cup N) \setminus V(G^\dagger)]$ contains a green connected-matching on all but at most one of its vertices. Thus, provided $k \geq 1/\eta^{1/2}$, there exists a connected-green matching G^\ddagger on at least $(\alpha_3 - 20\eta^{1/2})k$ vertices in $G[(M \cup N) \setminus V(G^\dagger)]$. Since G is $4\eta^4 k$ -almost-complete and all edges present in $G[M \cup N]$ are coloured exclusively green, all vertices of $M \cup N$ belong to the same green component of G . Thus, together, G^\dagger and G^\ddagger form a green connected-matching on at least $\alpha_3 k$ vertices. Thus, we may, after discarding at most $10\eta^{1/2}k$ vertices from each of $M \cup N$ and $P \cup Q$, assume that there are no green edges in $G[M \cup N, P \cup Q]$.

In summary, we now have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 218\eta)k, \quad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 142\eta)k, \quad |M \cup N| \geq (\alpha_3 - 20\eta^{1/2})k,$$

and know that all edges present in $G[M \cup N, P_1 \cup Q_1]$ are coloured exclusively red, all edges present in $G[M \cup N, P_2 \cup Q_2]$ are coloured exclusively blue and all edges in $G[M \cup N]$ are coloured exclusively green.

Thus, we have found, as a subgraph of G , a graph belonging to

$$\mathcal{K} \left((\tfrac{1}{2}\alpha_1 - 10\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 10\eta^{1/2})k, (\alpha_3 - 20\eta^{1/2})k, 4\eta^4 k \right),$$

completing Case E.iii.a.i.

Case E.iii.a.ii: P_2 and Q_2 belong to the different components.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ satisfying (E1)–(E3) such that

$$(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3 k \quad (\text{E4a})$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b})$$

Recall also that F , the largest monochromatic component in $P \cup Q$, is red and spans at least $|P \cup Q| - 8\eta k$ vertices and that each of M, N, P and Q have been subdivided into two parts such that $M_1 = M \cap V(R)$, $M_2 = M \setminus M_1$, $N_1 = N \cap V(R)$, $N_2 = N \setminus N_1$, $P_1 = P \cap V(R)$, $P_2 = P \setminus P_1$, $Q_1 = Q \cap V(R)$, $Q_2 = Q \setminus Q_1$, where R is the largest red matching in $G[M, Q] \cup G[N, P]$. By maximality of R , all edges present in $G[M_2, Q_2]$ or $G[N_2, P_2]$ are blue.

Additionally, in this case, we assume that P_2 and Q_2 belong to different blue components of G . Thus, in particular, all edges present in $G[N_2, Q_2]$ and $G[P_2, Q_2]$ are coloured red.

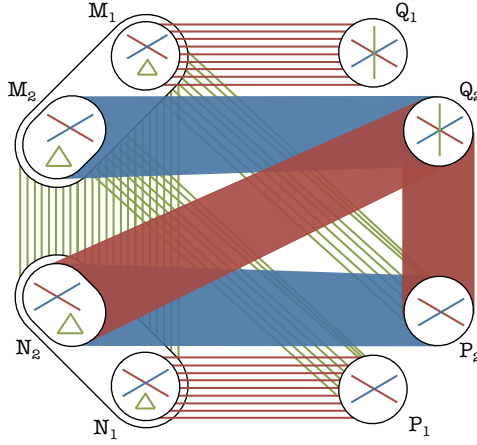


Figure 28: Initial colouring in Case E.iii.a.ii.

By Lemma 2.8, there exist red connected-matchings

R_{PQ} on at least $2 \min\{|P_2|, |Q_2|\} - 2\eta k$ vertices in $G[P_2, Q_2]$

R_{NQ} on at least $2 \min\{|N_2|, |Q_2|\} - 2\eta k$ vertices in $G[N_2, Q_2]$.

Then, since F includes all but at most $8\eta k$ of the vertices of $P \cup Q$, these connected-matchings can be augmented with edges from R to give the red connected-matchings illustrated in Figure 29:

R_1 on at least $2|P_1| + 2|Q_1| + 2 \min\{|P_2|, |Q_2|\} - 18\eta k$ vertices

in $G[M_1, Q_1] \cup G[N_1, P_1] \cup G[P_2, Q_2]$,

R_2 on at least $2|P_1| + 2|Q_1| + 2 \min\{|N_2|, |Q_2|\} - 18\eta k$ vertices

in $G[M_1, Q_1] \cup G[N_1, P_1] \cup G[N_2, Q_2]$,

Given the existence of these matchings, we can obtain bounds on the sizes of the various sets identified:

Since $|P|, |Q| \geq 95\eta^{1/2}k$, by (E4b), $|P|, |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 90\eta^{1/2})k$. But, then,

$$|M|, |N| \geq (\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2 - 5\eta^{1/2})k \geq \max\{|P|, |Q|\} + 85\eta^{1/2}k.$$

So, since $|M_1| = |Q_1|$ and $|N_1| = |P_1|$, we have

$$|M_2| \geq |Q_2| + 85\eta^{1/2}k, \quad |N_2| \geq |P_2| + 85\eta^{1/2}k. \quad (43)$$

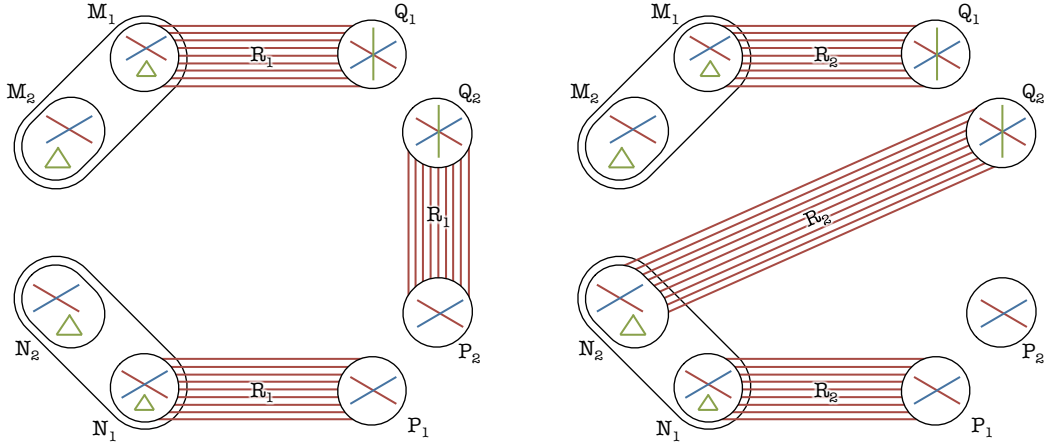


Figure 29: The red connected-matchings R_1 and R_2 .

In particular, $|M_2| \geq |Q_2|$ and $|N_2| \geq |P_2|$. Thus, by Lemma 2.8, we may obtain a blue connected-matching on $2|P_2| - 2\eta k$ vertices in $G[N_2, P_2]$ and one on $2|Q_2| - 2\eta k$ vertices in $G[M_2, Q_2]$. Thus, in order to avoid a blue connected-matching on at least $\alpha_2 k$ vertices, we may assume that

$$|P_2|, |Q_2| \leq (\frac{1}{2}\alpha_2 + \eta)k. \quad (44)$$

Suppose, for now, that $|Q_2| \geq |N_2|$. Then, recalling, that $|N_2| \geq |P_2|$, we have $|Q_2| \geq |P_2|$ and, therefore, R_1 spans at least $2|P_1| + 2|Q_1| + 2|P_2| - 18\eta k$ vertices in $G[M_1, Q_1] \cup G[N_1, P_1] \cup G[P_2, Q_2]$. Thus, in order to avoid a red connected-matching on at least $\alpha_1 k$ vertices, we may assume that

$$|P_1| + |Q_1| + |P_2| \leq (\frac{1}{2}\alpha_1 + 10\eta)k.$$

Also, since $|Q_2| \leq (\frac{1}{2}\alpha_2 + \eta)k$, by (E4b), we have

$$|P_1| + |Q_1| + |P_2| \geq (\frac{1}{2}\alpha_1 - 2\eta)k.$$

Thus, R_1 spans at least $(\alpha_1 - 22\eta)k$ vertices. Now, since $|Q_2| \geq |N_2|$, by (43), we have $|Q_2| \geq |P_2| + 15\eta^{1/2}k$. Thus, there exists $\tilde{Q} \subseteq Q_2 \setminus V(R_1)$ such that $|\tilde{Q}| \geq 15\eta^{1/2}k$. Therefore, by Lemma 2.8, we may find a red connected-matching M_3 on at least $28\eta^{1/2}k$ vertices in $G[N_2, \tilde{Q}]$ which belongs to the same red component as R_1 . Thus, together R_1 and R_3 form a red-connected-matching on $\alpha_1 k$ vertices, completing the proof in this case.

Therefore, we may instead assume that $|N_2| \geq |Q_2|$. In that case, R_2 spans at least $2|P_1| + 2|Q_1| + |Q_2| - 20\eta k$ vertices in $G[N_1, P_1] \cup G[M_1, Q_1] \cup G[N_2, Q_2]$. Thus, $|P_1| + |Q_1| + |Q_2| \leq (\frac{1}{2}\alpha_1 + 10\eta)k$.

Then, by (E4b) and (44), we obtain

$$\left. \begin{aligned} (\tfrac{1}{2}\alpha_1 - 2\eta)k &\leq |P_1| + |Q_1| + |Q_2| \leq (\tfrac{1}{2}\alpha_1 + 10\eta)k, \\ (\tfrac{1}{2}\alpha_1 - 12\eta)k &\leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta)k. \end{aligned} \right\} \quad (45)$$

Observe that, by (44) and (45), we may assume that

$$|Q_2| \leq |P_2| + \eta^{1/2}k. \quad (46)$$

We are now in a position to examine the coloured structure of $G[N, P]$. We show that, after possibly discarding some vertices, we may assume that all edges contained in $G[N, P_1]$ are coloured exclusively red and all edges contained in $G[N, P_2]$ are coloured exclusively blue. Following the same steps as in the proofs of Claim 3.1 and Claim 3.2 we prove:

Claim 3.5. *We may discard at most $67\eta k$ vertices from N_1 , $14\eta k$ vertices from N_2 , $54\eta k$ vertices from P_1 and at most $27\eta k$ vertices from P_2 such that, in what remains, all edges present in $G[N, P_1]$ are coloured exclusively red and all edges present in $G[N, P_2]$ are coloured exclusively blue.*

Proof. Given any $\tilde{N}_2 \subseteq N$ such that $|\tilde{N}_2| \geq |P_2| \geq (\tfrac{1}{2}\alpha_2 - 12\eta)k$, by Lemma 2.8, we can obtain a blue connected-matching on $(\alpha_2 - 26\eta)k$ vertices in $G[\tilde{N}_2, P_2]$. Then, since $|N_2| \geq |P_2| + 15\eta^{1/2}k$, the existence a blue matching B_S on at least $28\eta k$ vertices $G[N_2, P_1]$ would allow us to obtain a blue connected-matching on at least $\alpha_2 k$ vertices.

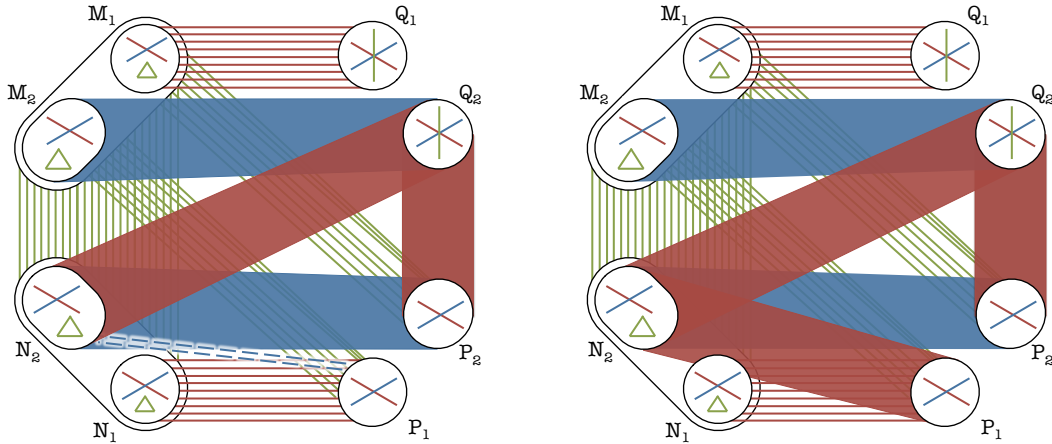


Figure 30: Colouring of the edges of $G[N_2, P_1]$.

Thus, after discarding at most $14\eta k$ vertices from each of P_1 and N_2 , we may assume that all edges present in $G[N_2, P_1]$ are coloured exclusively red.

After discarding these vertices, we have

$$|P_1| + |Q_1| + |Q_2| \geq (\tfrac{1}{2}\alpha_1 - 16\eta)k$$

and, thus, may assume that R_2 spans at least $(\alpha_1 - 50\eta)k$ vertices in $G[M_1, Q_1] \cup G[N_1, P_1] \cup G[N_2, Q_2]$.

Also, recalling (43) and (46), we have

$$|N_2| \geq |P_2| + 14\eta^{1/2}k \geq |Q_2| + 13\eta^{1/2}k.$$

Thus,

$$|N_2 \setminus V(R_2)| \geq |N_2| - |Q_2| \geq 13\eta^{1/2}k.$$

Suppose there exists a matching R_S on $54\eta k$ vertices in $G[N_1, P_2]$, then we can obtain a red connected-matching on at least $\alpha_1 k$ vertices as follows:

Observe that there exists a set R^- of $27\eta k$ edges belonging to R_2 such that $N_1 \cap V(R_S) = N_1 \cap V(R^-)$. Define $R^* = R_2 \setminus R^-$ and $\tilde{P} = P_1 \cap V(R^-)$, let \tilde{N} be any set of $27\eta k$ vertices in $N_2 \setminus V(R_2)$ and consider $G[\tilde{N}, \tilde{P}]$. Since $|\tilde{N}|, |\tilde{P}| \geq 27\eta k$, we may apply Lemma 2.8 to find a red connected-matching R_T on at least $52\eta k$ vertices in $G[\tilde{N}, \tilde{P}]$. Since all edges present in $G[N_2, Q_2] \cup G[P_2, Q_2]$ are coloured exclusively red, R_S and R_T belong to the same red component as R_2 . Then, $R^* \cup R_S \cup R_T$ is a red connected-matching in

$$G[M_1, Q_1] \cup G[N_1, P_1] \cup G[N_2, Q_2] \cup G[N_2, P_1] \cup G[N_1, P_2]$$

on at least $(\alpha_1 - 50\eta - 55\eta)k + 54\eta k + 52\eta k \geq \alpha_1 k$ vertices.

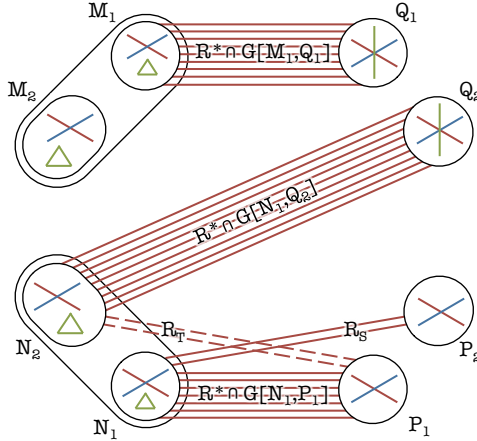


Figure 31: Enlarging the matching R_2 with edges from $G[N_1, P_2]$.

Thus, after discarding at most $27\eta k$ vertices from each of N_1 and P_2 , we may assume that all edges present in $G[N_1, P_2]$ are coloured exclusively blue and, recalling (45), that $|P_2| \geq (\frac{1}{2}\alpha_2 - 38\eta)k$.

Finally, suppose that there exists a blue matching B_S on at least $80\eta k$ vertices in $G[N_1, P_1]$. Then, we could obtain a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $40\eta k$ vertices from each of N_1 and P_1 , we may assume that all edges present in $G[N_1, P_1]$ are coloured exclusively red, thus completing the proof of the claim. \square

In summary, recalling (45), we now have

$$\begin{aligned} |P_1| + |Q_1| + |Q_2| &\geq (\tfrac{1}{2}\alpha_1 - 56\eta)k, \\ |P_2| &\geq (\tfrac{1}{2}\alpha_2 - 40\eta)k, \end{aligned}$$

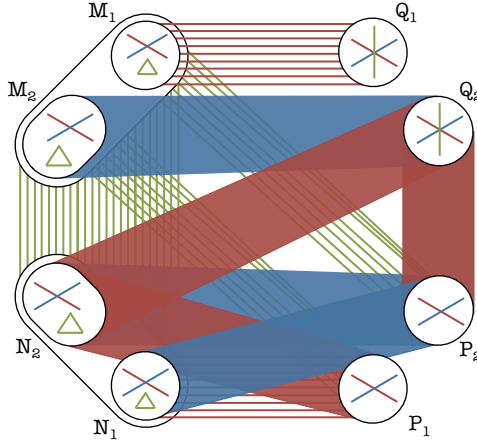


Figure 32: Resultant colouring of the edges of $G[N_1, P_2]$.

and know that all edges present in $G[N, P_1]$ are coloured exclusively red and that all edges present in $G[N, P_2]$ are coloured exclusively blue. Observe, also, that there can be no blue edges present in $G[N_1, Q_2]$ since then $M_2 \cup Q_2$ and $N_2 \cup P_2$ would belong to the same blue component of G .

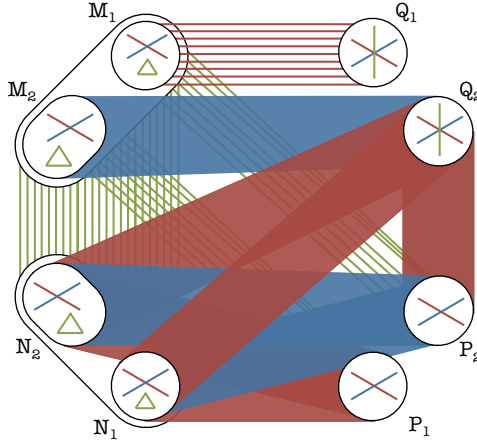


Figure 33: Colouring after Claim 3.5.

Next, we consider, in turn $G[M_2, N \cup P_2]$, $G[N]$ and $G[M_1, N]$ showing that after discarding a few vertices, we may assume that all edges remaining in each are coloured exclusively green:

Suppose there exists a red matching R_M on at least $136\eta k$ vertices in $G[M_2, N \cup P_2]$. Then, since $|N| \geq |P_1| + |Q_2| + 12\eta^{1/2}k$, we have $|N \setminus (R_M)| \geq |P_1| + |Q_2|$. Thus, by Lemma 2.8, there exists a red connected-matching R_N on at least $(2|P_1| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices in $G[N, P_1 \cup Q_2]$ sharing no vertices with R_M . Since all edges present in $G[N \cup P_2, Q_2] \cup G[N, P_1]$ are coloured red, R_M and R_N belong to the same red component of G .

Since $P \cup Q$ has a red effective-component on F on at least $|P \cup Q| - 8\eta k$ vertices, all but at most $8\eta k$ of the edges of R contained in $G[M_1, Q_1]$ belong to the same red component as $R_M \cup R_N$. Thus, defining R_Q to be the subset of R belonging to $G[M, Q_1 \cap F]$, we have a red connected-matching $R_M \cup R_N \cup R_Q$ on at least $136\eta k + (2|P_1| - 2\eta k) + (2|Q_2| - 2\eta k) + (2|Q_1| - 16\eta k) \geq \alpha_1 k$ vertices in $G[M_2, N \cup P_2] \cup G[N, P_1 \cup Q_2] \cup G[M_1, Q_1]$.

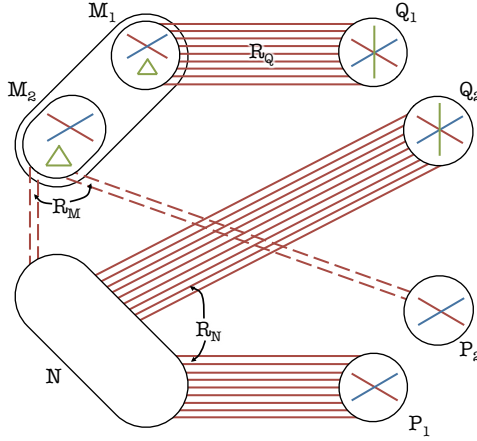


Figure 34: Construction of red connected-matching $R_M \cup R_N \cup R_Q$.

Thus, discarding at most $68\eta k$ vertices from each of M_2 and $N \cup P_2$, we may assume that there are no red edges $G[M_2, N \cup P_2]$. Thus, recalling that we assume that P_2 and Q_2 are in different blue components, all edges present in $G[M_2, N \cup P_2]$ are coloured exclusively green and have

$$\left. \begin{aligned} |N| &\geq |P_1| + |Q_2| + 11\eta^{1/2}k, & |P_1| + |Q_1| + |Q_2| &\geq (\tfrac{1}{2}\alpha_1 - 57\eta)k, \\ |N| &\geq |P_2| + 11\eta^{1/2}k, & |P_2| &\geq (\tfrac{1}{2}\alpha_2 - 108\eta)k. \end{aligned} \right\} \quad (47)$$

Next, suppose there exists a red matching R_A on $136\eta k$ vertices in $G[N]$. Then, by (47), we have $|N \setminus V(R_A)| \geq |P_1| + |Q_2|$. So, by Lemma 2.8, there exists a red connected-matching R_B on at least $(2|P_1| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices in $G[N \setminus V(R_A), P_1 \cup Q_2]$. Since all edges in $G[N, P_1]$ are coloured red, all edges of R_A and R_B belong to the same red component. Also, since the red component F spans all but at most $8\eta k$ vertices of $P \cup Q$, there exists a red-matching $R_C \subseteq R$ in $G[M_1, Q_1]$ on at least $2|Q_1| - 16\eta k$ vertices belonging to the same red component as R_A and R_B . Then, together R_A , R_B and R_C form a red connected-matching on at least $\alpha_1 k$ vertices in $G[N] \cup G[N, P_1 \cup Q_2] \cup G[M_1 \cup Q_1]$.

Similarly, if there exists a blue matching B_A on $218\eta k$ vertices in $G[N]$, then we can construct a blue connected-matching on at least $\alpha_2 k$ vertices as follows. By (47), we have $|N \setminus V(R_B)| \geq |P_2|$. So, by Lemma 2.8, there exists a blue connected-matching B_B on at least $2|P_2| - 2\eta k$ vertices in $G[N \setminus V(B_A), P_2]$. Since all edges in $G[N, P_2]$ are coloured blue, all edges in B_A and B_B belong to the same blue component. Thus, together, B_A and B_B form a blue connected-matching on at least $2|P_2| + 216\eta k \geq \alpha_2 k$ vertices in $G[N] \cup G[N, P_1 \cup Q_2] \cup G[M_1 \cup Q_1]$. Thus, discarding at most $354\eta k$ vertices from N , we have

$$|N| \geq |P_1| + |Q_2| + 10\eta^{1/2}k, \quad |N| \geq |P_2| + 10\eta^{1/2}k \quad (48)$$

and may assume that all edges in $G[N]$ are coloured exclusively green.

Finally, we consider $G[M_1, N]$. Suppose there exists a red matching R_D on $138\eta k$ vertices in $G[M_1, N]$. Then, by (48), we have $|N \setminus V(R_D)| \geq |P_1| + |Q_2|$. Therefore, by Lemma 2.8, there exists a red connected-matching R_E on at least $(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k)$ vertices in $G[N, P_1 \cup Q_2]$ which shares no vertices with R_D . Then, since all edges present in $G[N, P_1]$ are coloured red, R_D and R_E belong to the same red component. Since F , the largest red component in $G[P \cup Q]$ includes all

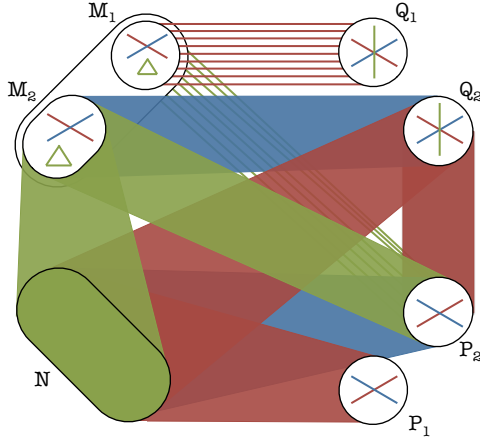


Figure 35: Colouring of $G[M_2, P_2] \cup G[N]$.

but at most $8\eta k$ of the vertices of $P \cup Q$, there exists a matching $R_F \subseteq R$ in $G[M_1, Q_1]$ on at least $2(|Q_1| - |M_1 \cap V(R_D)| - 8\eta k)$ vertices which shares no vertices with R_D but belongs to the same red component as it. Thus, together, R_D , R_E and R_F form a red connected-matching on at least $2(|P_1| + |Q_1| + |Q_2| + |N \cap V(R_D)|) - 20\eta k \geq \alpha_1 k$ vertices.

Similarly, if there exists a blue matching B_D on $220\eta k$ vertices in $G[M_1, N]$. Then, by (48), we have $|N \setminus V(B_D)| \geq |P_2|$. Therefore, by Lemma 2.8, there exists a blue connected-matching B_E on at least $(2|P_2| - 2\eta k)$ vertices in $G[N, P_2]$ which shares no vertices with B_D . Since all edges present in $G[N, P_2]$ are coloured blue, B_D and B_E belong to the same blue component. Thus together B_D and B_E form a blue connected-matching on at least $2|P_2| - 2\eta k + 220\eta k \geq \alpha_2 k$ vertices. Therefore, after discarding at most $174\eta k$ vertices from each of M_1 and N , we may assume that all edges present in $G[M_1, N]$ are coloured exclusively green.

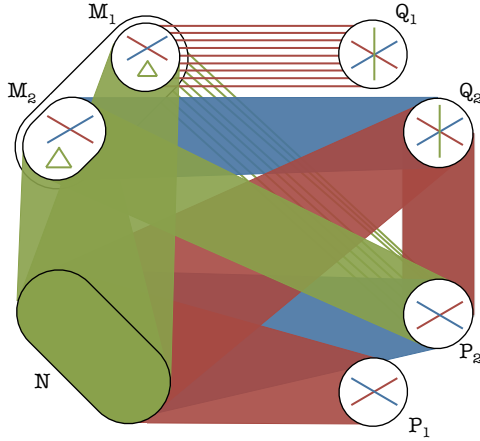


Figure 36: Final Colouring in Case E.iii.a.ii.

Given the colouring found so far, we now show that we may obtain a green connected-matching on at least $\alpha_3 k$ vertices: Having discarded at most $1000\eta k$ vertices from $M \cup N$, recalling (E4a), since $\eta < 10^{-7}$, we have $|M|, |N| \geq (\frac{1}{2}\alpha_3 - 6\eta^{1/2})k$. Recalling (43), we have $|M_2| \geq |Q_2| + 14\eta^{1/2}k \geq 14\eta^{1/2}$ and, by (47), have $|P_2| \geq (\frac{1}{2}\alpha_2 - 108\eta)k \geq 14\eta^{1/2}k$. Letting M' be a subset of M_2 and N' a subset of

N such that $14\eta^{1/2}k \leq |M'| = |N'| \leq 15\eta^{1/2}k$, by Lemma 2.8, there exist green matchings G_{MP} on at least $2|M'| - 2\eta k$ vertices in $G[M', P_2]$ and G_{MN} on at least $2\min\{|M \setminus M'|, |N \setminus N'|\} - 2\eta k$ vertices in $G[M \setminus M', N \setminus N']$. Finally, by Theorem 2.1, provided $k \geq 1/\eta^2$, there exists a connected-matching G_N on at least $|N'| - 1$ vertices in $G[N']$. Then, since all edges present in $G[M, N]$ are coloured green, G_{NP} , G_{MN} and G_N belong to the same green component and, since they share no vertices, form a green connected-matching on at least

$$(2|M'| - 2\eta k) + (2\min\{|M \setminus M'|, |N \setminus N'|\} - 2\eta k) + |N'| - 1 \\ \geq \alpha_3 k + \eta^{1/2}k - 4\eta k - 1 \geq \alpha_3 k$$

vertices. By the definition of the decomposition $M \cup N \cup P \cup Q$, this connected-matching is odd, thus completing Case E.iii.a.ii.

At the beginning of Case E.iii.a, we made the assumption that F , the largest monochromatic component in $G[P \cup Q]$ was red. If, instead, F is blue, then the proof is essentially identical to the above with the roles of red and blue reversed. The result is the same, that is, G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K}\left(\left(\frac{1}{2}\alpha_1 - 10\eta^{1/2}\right)k, \left(\frac{1}{2}\alpha_1 - 10\eta^{1/2}\right)k, (\alpha_3 - 20\eta^{1/2})k, 4\eta^4 k\right),$$

thus completing Case E.iii.a.

Case E.iii.b: $G[P, Q]$ contains red and blue stars centred in Q .

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ satisfying (E1)–(E3) with $|P|, |Q| \geq 95\eta^{1/2}k$ and

$$(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3 k \quad (\text{E4a})$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b})$$

Additionally, in this case, we assume that the sets

$$W_r = \{q \in Q : q \text{ has red edges to all but at most } 8\eta k \text{ vertices in } P\},$$

$$W_b = \{q \in Q : q \text{ has blue edges to all but at most } 8\eta k \text{ vertices in } P\},$$

are both non-empty.

We define two further sets which will be useful in what follows:

$$P_r = \{p \in P : p \text{ has a red edge to some vertex } q \in W_r\},$$

$$P_b = \{p \in P : p \text{ has a blue edge to some vertex } q \in W_b\}.$$

Observe that, since $|P|, |Q| \geq 95\eta^{1/2}k$, by (E4b), we have

$$95\eta^{1/2}k \leq |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 90\eta^{1/2})k.$$

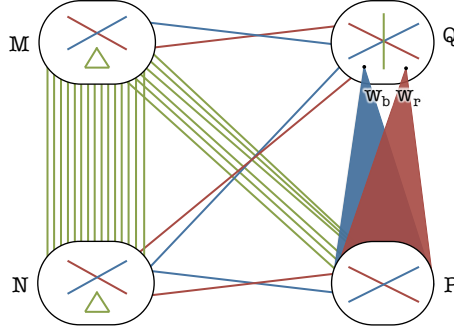


Figure 37: Red and blue stars centred in Q .

Thus, considering (E4a) and (E4b), we have

$$|N| \geq |Q| \geq 95\eta^{1/2} \geq 6(2\eta^{1/2})|N \cup Q| \geq 6(2\eta)|N \cup Q|.$$

Recall that, at the start of Case E, after discarding some edges, G was assumed to be $(1 - \frac{3}{2}\eta^4)$ -complete. Recall also that, from (11), we have $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}$. Thus, considering (E4a), since $|Q| \geq 95\eta^{1/2}k$, we have

$$|N \cup Q| \geq \frac{1}{2}\alpha_3 k \geq \frac{1}{4}(\alpha_3 + \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2})k \geq \frac{1}{4}(\alpha_3 + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 10\eta^{1/2})k \geq \frac{1}{4}K.$$

Then, since $\eta < 0.1$, we have $6\eta^4 \leq 2\eta$ so $G[N \cup Q]$ is $(1 - 2\eta)$ -complete and, provided $|N \cup Q| \geq 1/\eta$, we may apply Lemma 2.13 to $G[N, Q]$ and distinguish four cases:

- (i) $G[N \cup Q]$ has a monochromatic component E on at least $|N \cup Q| - 28\eta k$ vertices;
- (ii) N, Q can be partitioned into $N_1 \cup N_2, Q_1 \cup Q_2$ such that $|N_1|, |N_2|, |Q_1|, |Q_2| \geq 3\eta k$ and all edges present between N_i and Q_j are red for $i = j$ and blue for $i \neq j$;
- (iii) there exist vertices $n_r, n_b \in N$ such that n_r has red edges to all but $16\eta k$ vertices in Q and n_b has blue edges to all but $16\eta k$ vertices in Q ;
- (iv) there exist vertices $q_r, q_b \in Q$ such that q_r has red edges to all but $16\eta k$ vertices in N and q_b has blue edges to all but $16\eta k$ vertices in N .

Case E.iii.b.i: $G[N \cup Q]$ has a large monochromatic component.

Suppose that E , the largest monochromatic component in $G[N \cup Q]$, is red. In that case, if $G[Q \cap E, P_r]$ contains a red edge, then $G[P \cup Q]$ has a red effective-component on at least $|P \cup Q| - 36\eta k$ vertices. Alternatively, every edge in $G[Q \cap E, P_r]$ is blue, in which case, $G[P \cup Q]$ has a blue connected-component on at least $|P \cup Q| - 36\eta k$ vertices.

Suppose instead that E is blue. In that case, if $G[Q \cap E, P_b]$ contains a blue edge, then $G[P \cup Q]$ has a blue effective-component on at least $|P \cup Q| - 36\eta k$ vertices. Alternatively, every edge present in $G[Q \cap E, P_b]$ is red, in which case, $G[P \cup Q]$ has a red connected-component on at least $|P \cup Q| - 36\eta k$ vertices.

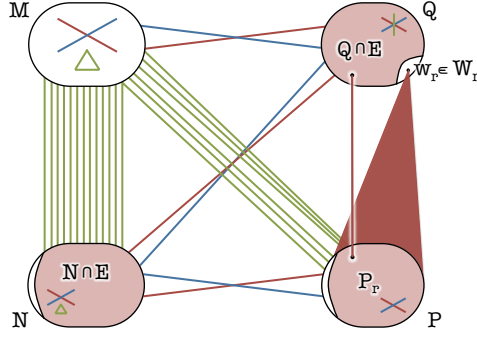


Figure 38: Large red effective-component.

In either case the proof proceeds via exactly the same steps as Case E.iii.a with the result being that G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K} \left(\left(\frac{1}{2}\alpha_1 - 100\eta^{1/2} \right)k, \left(\frac{1}{2}\alpha_1 - 100\eta^{1/2} \right)k, (\alpha_3 - 200\eta^{1/2})k, 4\eta^4 k \right),$$

thus completing Case E.iii.b.i.

Case E.iii.b.ii: $N \cup Q$ has a non-trivial partition with ‘cross’ colouring.

In this case, we assume that N and Q can be partitioned into $N_1 \cup N_2$, $Q_1 \cup Q_2$ such that $|N_1|, |N_2|, |Q_1|, |Q_2| \geq 3\eta k$ and all edges present between N_i and Q_j are red for $i = j$ and blue for $i \neq j$.

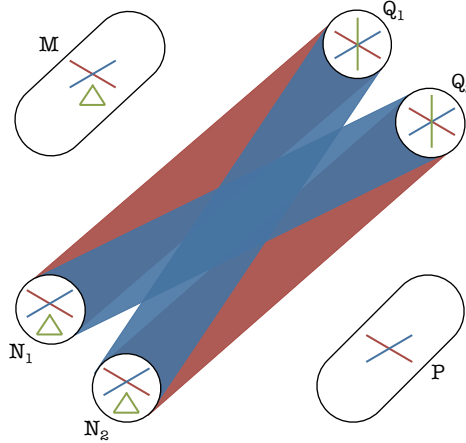


Figure 39: ‘Cross’ colouring of $G[N, Q]$.

Recall that W_r and W_b are both non-empty, that is, there exist vertices $w_r, w_b \in Q$ such that w_r has red edges to all but $8\eta k$ vertices in P and w_b has blue edges to all but $8\eta k$ vertices in P .

Observe that, we may assume that $P \cup Q$ does not have a monochromatic effective-component on at least $|P \cup Q| - 16\eta k$ vertices. Indeed, otherwise, the proof proceeds via the same steps as Case E.iii.a, with the result being that G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$

vertices or a subgraph in

$$\mathcal{K} \left((\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\alpha_3 - 200\eta^{1/2})k, 4\eta^4 k \right).$$

Without loss of generality, we assume $w_r \in Q_1$. Observe then that the existence of a red edge in $G[Q_2, P_r]$ would result in $P \cup Q$ having a red effective-component on at least $|P \cup Q| - 8\eta k$ vertices. Thus, we assume that every edge present in $G[Q_2, P_r]$ is blue. Similarly, we may assume every edge present in $G[Q_1, P_b]$ is red. Since W_r and W_b are non-empty, $|P_r|, |P_b| \geq |P| - 8\eta k$. Then, as $|P| \geq 95\eta^{1/2}k$, we have $|P_r \cap P_b| \geq |P| - 16\eta k > 0$ and know that all edges present in $G[Q_2, P_r \cap P_b]$ are coloured exclusively red and all edges present in $G[Q_1, P_r \cap P_b]$ are coloured exclusively blue.

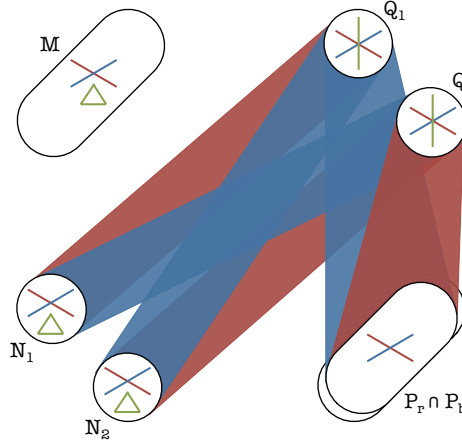


Figure 40: Red and blue edges in $G[N, Q] \cup G[P, Q]$.

Recall that, by (E3), there are no green edges in $G[P, N]$ and notice that, if there existed a vertex $p \in P$ with red edges to both N_1 and N_2 , then $P \cup Q$ would have an effective red-component on at least $|P \cup Q| - 16\eta k$ vertices. Thus, we assume that there is no such vertex. Similarly, we assume there does not exist a vertex in P with blue edges to both N_1 and N_2 . Thus, P can be partitioned into $P_1 \cup P_2$ such that all edges present between P_i and N_j are red for $i = j$ and blue for $i \neq j$.

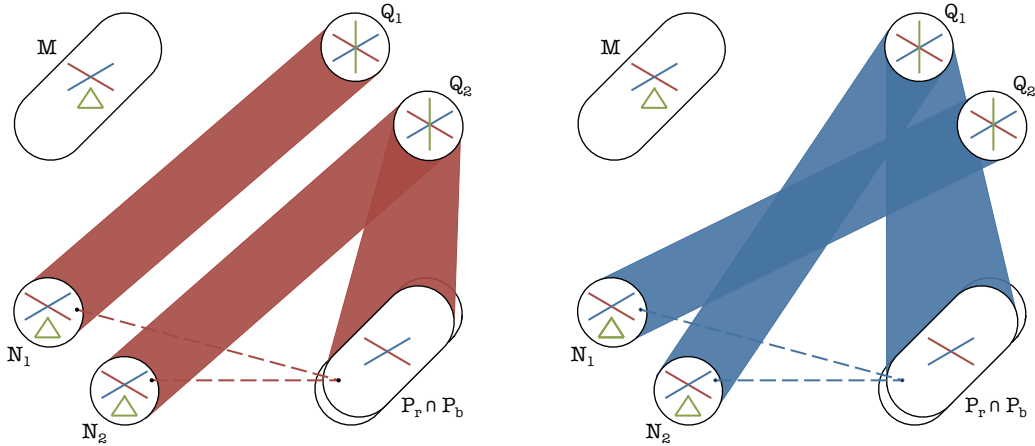


Figure 41: Partitioning of P .

Recall, however, that all edges present in $G[Q_1, P_r]$ are red and that all edges present in $G[Q_2, P_b]$ are

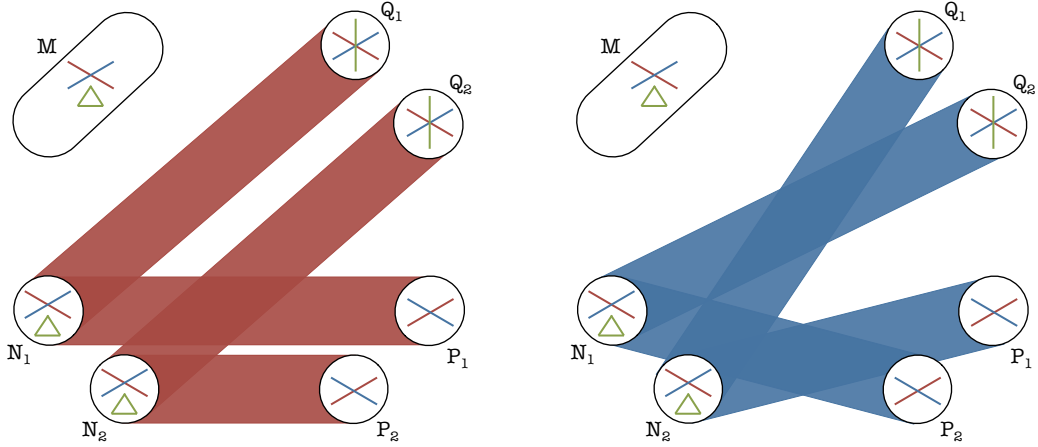


Figure 42: Resultant colouring of $G[N, P]$.

blue. Thus, in order to avoid $G[P, Q]$ having a red effective-component on at least $|P \cup Q| - 16\eta k$ vertices, we must have $P_r \subseteq P_1$ but then, since $|P_r \cap P_b| > 0$, there exists a blue edge in $G[Q_2, P_1]$, giving rise to a blue effective-component on at least $|P \cup Q| - 16\eta k$ vertices, completing Case E.iii.a.ii.

Case E.iii.b.iii: $G[N, Q]$ contains red and blue stars centred in N .

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ satisfying (E1)–(E4b) and that there exists vertices $w_r \in W_r$ and $w_b \in W_b$, where

$$W_r = \{q \in Q : q \text{ has red edges to all but at most } 8\eta k \text{ vertices in } P\},$$

$$W_b = \{q \in Q : q \text{ has blue edges to all but at most } 8\eta k \text{ vertices in } P\}.$$

Additionally, in this case, we assume that there exist vertices $n_r \in N_r$ and $n_b \in N_b$, where

$$N_r = \{n \in N : n \text{ has red edges to all but at most } 16\eta k \text{ vertices in } Q\},$$

$$N_b = \{n \in N : n \text{ has blue edges to all but at most } 16\eta k \text{ vertices in } Q\}.$$

Recall that

$$P_r = \{p \in P : p \text{ has a red edge to some vertex } q \in W_r\},$$

$$P_b = \{p \in P : p \text{ has a blue edge to some vertex } q \in W_b\},$$

and define $Q_N = \{q \in Q : q \text{ has a red edge to some vertex } n \in N_r\}$.

Notice that, since $|W_r|, |W_b| > 0$, we have $|P_r|, |P_b| \geq |P| - 8\eta k$ and, since $|N_r| > 0$, we have $|Q_N| \geq |Q| - 16\eta k$. So, if there exists a red edge in $G[P_r, Q_N]$, then $G[P \cup Q]$ has a red effective-component on at least $|P \cup Q| - 24\eta k$ vertices. Alternatively, every edge present in $G[P_r, Q_N]$ is blue. Then $G[P \cup Q]$ has a blue component on at least $|P \cup Q| - 24\eta k$ vertices. In either case, the proof then follows the same steps as in Case E.iii.a with the result being that G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd

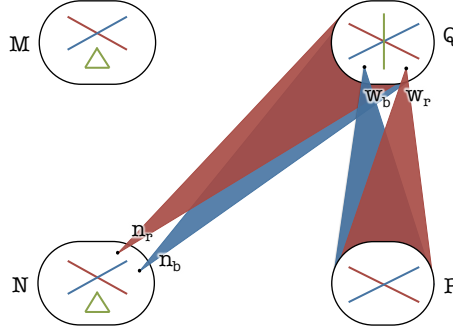


Figure 43: ‘Stars’ centred at w_r, w_b, n_r and n_b .

connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K} \left(\left(\frac{1}{2}\alpha_1 - 100\eta^{1/2} \right)k, \left(\frac{1}{2}\alpha_1 - 100\eta^{1/2} \right)k, (\alpha_3 - 200\eta^{1/2})k, 4\eta^4 k \right),$$

thus completing case E.iii.b.iii.

Case E.iii.b.iv: $G[N, Q]$ contains red and blue stars centred in Q .

Recall that we have a decomposition of $V(G)$ into four parts $M \cup N \cup P \cup Q$ satisfying

- (E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;
- (E2) every vertex in P has a green edge to M ;
- (E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$;

such that the sizes of the four parts satisfy

$$(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|, |N| \leq \frac{1}{2}\alpha_3 k, \quad (\text{E4a})$$

$$(\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k. \quad (\text{E4b})$$

Recall, also, that there exists vertices $w_r \in W_r$ and $w_b \in W_b$, where

$$W_r = \{q \in Q : q \text{ has red edges to all but at most } 8\eta k \text{ vertices in } P\},$$

$$W_b = \{q \in Q : q \text{ has blue edges to all but at most } 8\eta k \text{ vertices in } P\}.$$

Additionally, in this case, we assume that there exist vertices $q_r \in Q_r$ and $q_b \in Q_b$, where

$$Q_r = \{n \in Q : n \text{ has red edges to all but at most } 16\eta k \text{ vertices in } N\},$$

$$Q_b = \{n \in Q : n \text{ has blue edges to all but at most } 16\eta k \text{ vertices in } N\}.$$

We consider $G[Q \cup M]$. Since all edges present in $G[Q, M]$ are coloured red or blue, Lemma 2.13 can be applied to $G[M \cup Q]$ as it was previously applied to $G[N \cup Q]$ with the same four possible outcomes:

- (i) $G[M \cup Q]$ has a monochromatic component on at least $|M \cup Q| - 28\eta k$ vertices;

- (ii) M, Q can be partitioned into $M_1 \cup M_2, Q_1 \cup Q_2$ such that $|M_1|, |M_2|, |Q_1|, |Q_2| \geq 3\eta k$ and all edges present between M_i and Q_j are red for $i = j$ and blue for $i \neq j$;
- (iii) there exist vertices $m_r, m_b \in N$ such that m_r has red edges to all but $16\eta k$ vertices in Q and m_b has blue edges to all but $16\eta k$ vertices in Q ;
- (iv) there exist vertices $q_r, q_b \in Q$ such that q_r has red edges to all but $16\eta k$ vertices in M and q_b has blue edges to all but $16\eta k$ vertices in M .

For possibilities (i)–(iii), the proof proceeds via exactly the same steps as in the corresponding cases above with the same possible outcomes. Thus, we consider possibility (iv). That is, we assume that (in addition to w_r, w_b, q_r, q_b) there exist vertices $v_r, v_b \in Q$ such that v_r has red edges to all but $16\eta k$ vertices in M and v_b has blue edges to all but $16\eta k$ vertices in M .

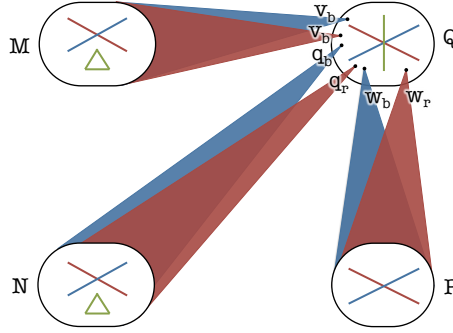


Figure 44: ‘Stars’ centred in Q .

Consider the largest red matching R in $G[N, P \cup Q]$ and partition each of N, P and Q into two parts such that $N_1 = N \cap V(R)$, $N_2 = N \setminus N_1$, $P_1 = P \cap V(R)$, $P_2 = P \setminus P_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \setminus Q_1$. By maximality of R , all edges present in $G[N_2, P_2 \cup Q_2]$ are coloured exclusively blue.

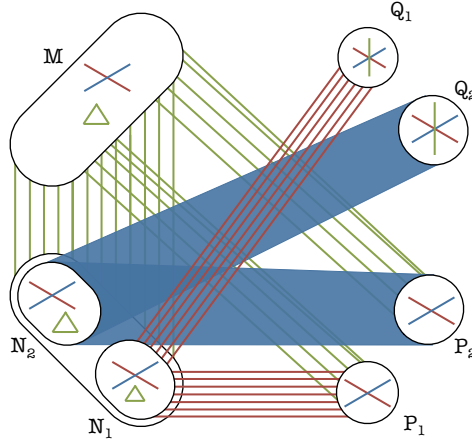


Figure 45: Partition of N, P and Q .

Notice that, in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices, by Lemma 2.8, we may assume that $\min\{|P_2 \cup Q_2|, |N_2|\} \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$. Thus, by (E4a) and (E4b), we may thus assume that

$$|N_1| = |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 6\eta^{1/2})k. \quad (49a)$$

Since q_r has red edges to all but at most $16\eta k$ vertices in N , all but at most $16\eta k$ of the edges of R belong to the same red component and we have a red connected-matching on $2(|N_1| - 16\eta k)$ vertices in $G[N_1, P_1 \cup Q_1]$. Thus, we may assume that

$$|N_1| = |P_1 \cup Q_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k$$

and, therefore, by (E4a) and (E4b), that

$$|N_2|, |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 6\eta^{1/2})k. \quad (49b)$$

Recall that v_r has red edges to all but $16\eta k$ vertices in M and v_b has blue edges to all but $16\eta k$ vertices in M . Thus, we may, after discarding at most $32\eta k$ vertices from M , assume that $G[M]$ is effectively red-connected and effectively blue-connected.

Similarly, we may, after discarding at most $16\eta k$ vertices from $G[P]$, assume that P is effectively red-connected and effectively blue-connected and, after discarding at most $32\eta k$ vertices from N , assume that N is effectively red-connected and effectively blue-connected. In order to maintain the equality $|N_1| = |P_1|$, we also discard from $N_1 \cup P_1$ any vertex whose R -mate has already been discarded.

Then, in summary, having discarded some vertices, we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ and a refinement into $M \cup N_1 \cup N_2 \cup P_1 \cup P_2 \cup Q_1 \cup Q_2$ such that

(E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$;

(E6a) $G[M]$, $G[N]$ and $G[P]$ each have a single red and a single blue effective-component;

(E6b) $G[N_1, P_1 \cup Q_1]$ contains a red matching utilising every vertex in $G[N_1 \cup P_1 \cup Q_1]$;

(E6c) all edges present in $G[N_2, P_2 \cup Q_2]$ are coloured exclusively blue.

Having discarded some vertices, recalling (E4a), (49a) and (49b), we have

$$\left. \begin{aligned} |M| &\geq (\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 6\eta^{1/2})k, & |N_1| = |P_1 \cup Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 8\eta^{1/2})k, \\ |N| &\geq (\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 7\eta^{1/2})k, & |N_2|, |P_2 \cup Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 8\eta^{1/2})k. \end{aligned} \right\} \quad (E7)$$

and also

$$|P| \geq 90\eta^{1/2}k, \quad \max\{|Q_1|, |Q_2|\} \geq 45\eta^{1/2}k. \quad (E8)$$

For the final time, we distinguish between three cases:

- (a) $G[M, N \cup P \cup Q_1]$ contains a red edge;
- (b) $G[M, N \cup P \cup Q_2]$ contains a blue edge;
- (c) $G_1[M, N \cup P \cup Q_1]$ and $G_2[M, N \cup P \cup Q_2]$ each contain no edges.

Case E.iii.b.iv.a: $G[M, N \cup P \cup Q_1]$ contains a red edge.

Given the existence of a red edge in $G[M, N \cup P \cup Q_1, M]$, then the existence of a red matching on at least $14\eta^{1/2}k$ vertices in $G[P_2 \cup Q_2, M]$ would give a red connected-matching on at least $\alpha_1 k$ vertices. Thus, we may, after discarding at most $7\eta^{1/2}k$ vertices from each of M and $P_2 \cup Q_2$, assume that all present edges in $G[M, Q_2]$ are coloured exclusively blue and that there are no red edges in $G[M, P_2]$.

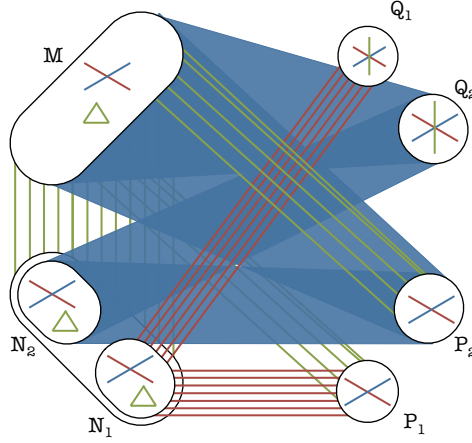


Figure 46: Colouring immediately before Claim 3.6.

We then have

$$\left. \begin{aligned} |N_1| = |P_1 \cup Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 8\eta^{1/2})k, \\ |N_2|, |P_2 \cup Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 15\eta^{1/2})k. \end{aligned} \right\} \quad (50)$$

The following pair of claims establish the coloured structure of $G[M \cup N, P \cup Q]$:

Claim 3.6.a. *If $|Q_1| \geq 45\eta^{1/2}k$, we may discard at most $43\eta^{1/2}k$ vertices from N_1 , at most $43\eta^{1/2}k$ vertices from N_2 , at most $16\eta^{1/2}k$ vertices from M , at most $59\eta^{1/2}k$ vertices from $P_1 \cup Q_1$ and at most $27\eta^{1/2}k$ vertices from $P_2 \cup Q_2$ such that, in what remains, there are no blue edges present in $G[M \cup N, P_1 \cup Q_1]$ and no red edges present in $G[M \cup N, P_2 \cup Q_2]$.*

Claim 3.6.b. *If $|Q_2| \geq 45\eta^{1/2}k$, we may discard at most $45\eta^{1/2}k$ vertices from N_1 , at most $18\eta^{1/2}k$ vertices from N_2 , at most $18\eta^{1/2}k$ vertices from M , at most $18\eta^{1/2}k$ vertices from $P_1 \cup Q_1$ and at most $27\eta^{1/2}k$ vertices from $P_2 \cup Q_2$ such that, in what remains, there are no blue edges present in $G[M \cup N, P_1 \cup Q_1]$ and no red edges present in $G[M \cup N, P_2 \cup Q_2]$.*

Proof. (a) We begin by considering $G[M \cup N_1, P_1 \cup Q_1]$. Suppose there exists a blue matching B_S on $32\eta^{1/2}k$ vertices in $G[M \cup N_1, P_1 \cup Q_1]$. Observe that, since $|N_2|, |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 15\eta^{1/2})k$, by Lemma 2.8, there exists a blue connected-matching B_L on at least $(\alpha_2 - 32\eta^{1/2})k$ vertices in $G[N_2, P_2 \cup Q_2]$. Then, since all edges present in $G[M \cup N_2, Q_2]$ are coloured blue and $G[N]$ is blue effectively-connected, B_S and B_L belong to the same blue component and thus form a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $16\eta^{1/2}k$ vertices from each of $M \cup N_1$ and $P_1 \cup Q_1$, we may assume that there are no blue edges present in $G[M \cup N_1, P_1 \cup Q_1]$. Notice then, in particular, that all edges present in $G[M, Q_1] \cup G[N_1, P_1 \cup Q_1]$ are coloured exclusively red.

Having discarding these vertices, we have

$$|Q_1| \geq 29\eta^{1/2}k, \quad |N_1| \geq |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 24\eta^{1/2})k.$$

Now, suppose there exists a red matching R_S on at least $54\eta^{1/2}k$ vertices in $G[N_1, P_2 \cup Q_2]$. Since $|P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 24\eta^{1/2})k$ and $|Q_2| \geq 29\eta^{1/2}k$, there exists $\tilde{Q} \subseteq Q_1$ such that $|P_1 \cup \tilde{Q}| \geq (\frac{1}{2}\alpha_1 - 52\eta^{1/2})k$

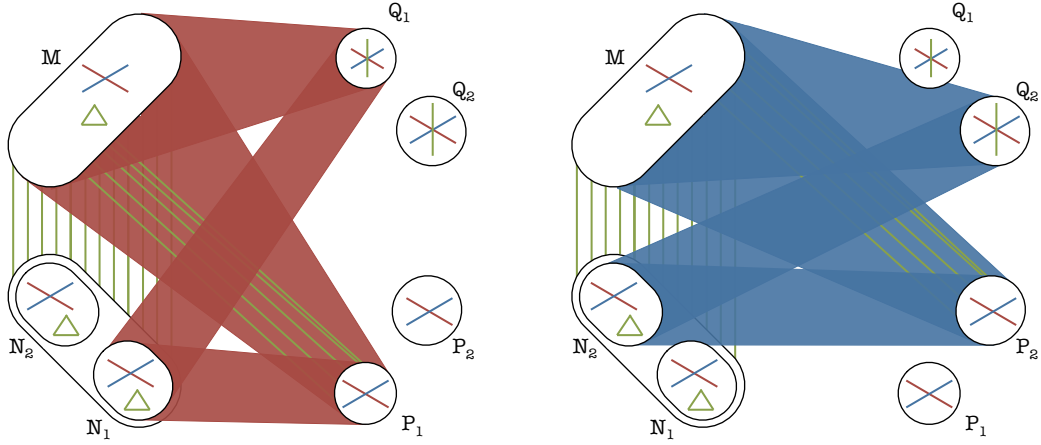


Figure 47: Colouring of the edges of $G[M \cup N_1, P_1 \cup Q_1]$ in Claim 3.6.a.

and $|Q_1 \setminus \tilde{Q}| \geq 27\eta^{1/2}k$. Then, since $|N_1 \setminus V(R_S)|, |P_1 \cup \tilde{Q}| \geq (\frac{1}{2}\alpha_1 - 52\eta^{1/2})k$ and $|M|, |Q_1 \setminus \tilde{Q}| \geq 27\eta^{1/2}k$, by Lemma 2.8, there exist red connected-matchings R_L on at least $(\alpha_1 - 106\eta^{1/2})k$ vertices in $G[N_1, P_1 \cup Q_1]$ and R_T on at least $52\eta^{1/2}k$ vertices in $G[Q_1, M]$ sharing no vertices with each other or R_S . Then, since all edges in $G[N_1, Q_1]$ are coloured red, R_L, R_S and R_T belong to the same red component and, thus, together, form a red connected-matching on at least $\alpha_1 k$ vertices. Therefore, after discarding at most $27\eta^{1/2}k$ vertices from each of N_1 and $P_2 \cup Q_2$, we may assume that all edges in $G[N_1, P_2 \cup Q_2]$ are coloured exclusively blue.

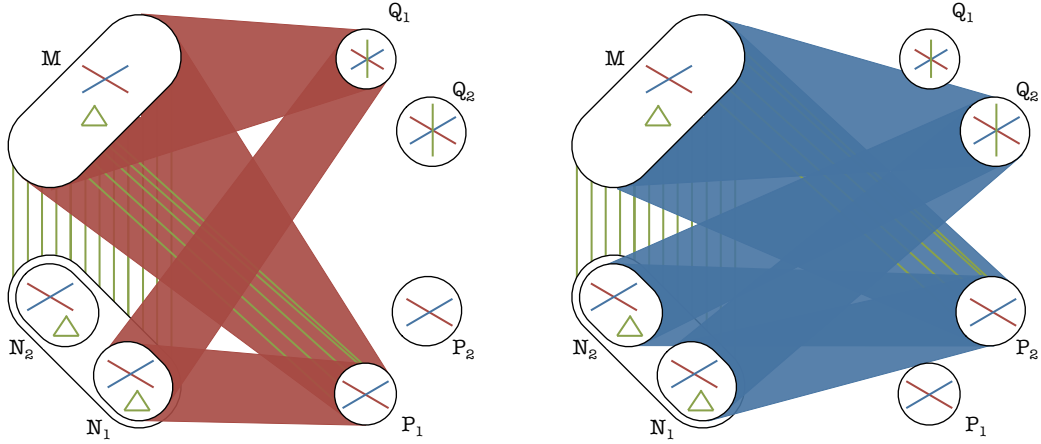


Figure 48: Colouring of the edges of $G[N_1, P_2 \cup Q_2]$ in Claim 3.6.a.

We then have

$$\begin{aligned} |N_1| &\geq (\tfrac{1}{2}\alpha_1 - 51\eta^{1/2})k, & |P_1 \cup Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 24\eta^{1/2})k, \\ |N_2| &\geq (\tfrac{1}{2}\alpha_2 - 15\eta^{1/2})k, & |P_2 \cup Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 42\eta^{1/2})k. \end{aligned}$$

Suppose now that there exists a blue matching B_U on $86\eta^{1/2}k$ vertices in $G[N_2, P_1 \cup Q_1]$. Then, since $|N \setminus V(B_U)|, |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 42\eta^{1/2})k$, by Lemma 2.8, there exists a blue connected-matching B_V on at least $(\alpha_2 - 86\eta^{1/2})k$ vertices in $G[N \setminus V(B_U), P_2 \cup Q_2]$. Therefore, $B_U \cup B_V$ forms a blue connected-matching on at least $\alpha_2 k$ vertices in $G[N, P \cup Q]$. Thus, after discarding at most $43\eta^{1/2}k$

vertices from each of N_2 and $P_1 \cup Q_1$, we may assume that all edges present in $G[N_1, P_1 \cup Q_1]$ are coloured exclusively red, thus completing the proof of Claim 3.6.a.

(b) Suppose that G is coloured as in Figure 46 and that $|Q_2| \geq 45\eta^{1/2}k$. We begin by considering $G[M \cup N, P_1 \cup Q_1]$. Suppose there exists a blue matching B_S on $36\eta^{1/2}k$ vertices in $G[M \cup N, P_1 \cup Q_1]$, then we can obtain a blue connected-matching on at least $\alpha_1 k$ vertices as follows: Recalling (50), we have $|P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 15\eta^{1/2})k$. Then, since $|Q_2| \geq 45\eta^{1/2}k$, there exists $\tilde{Q} \subseteq Q_2$ such that $|P_2 \cup \tilde{Q}| \geq (\frac{1}{2}\alpha_2 - 34\eta^{1/2})k$ and $|Q_2 \setminus \tilde{Q}| \geq 18\eta^{1/2}k$. Then, we have $|N_2 \setminus V(B_S)|, |P_2 \cup \tilde{Q}| \geq (\frac{1}{2}\alpha_2 k - 34\eta^{1/2})k$ and $|M \setminus V(B_S)|, |Q_2 \setminus \tilde{Q}| \geq 18\eta^{1/2}k$. Thus, by Lemma 2.8, there exist blue connected-matchings B_L on at least $(\alpha_2 k - 70\eta^{1/2})k$ vertices in $G[N \setminus V(B_S), P_2 \cup \tilde{Q}]$ and B_T on at least $34\eta^{1/2}k$ vertices in $G[M \setminus V(B_S), Q_2 \setminus \tilde{Q}]$. Since M and N are each blue effectively-connected and all edges present in $G[M \cup N_2, P_2]$ are coloured blue, B_L, B_S and B_T belong to the same blue component in G and, thus, together, form a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, after discarding at most $18\eta^{1/2}k$ vertices from each of $M \cup N$ and $P_1 \cup Q_1$, we may assume that all edges present in $G[M \cup N, P_1 \cup Q_1]$ are coloured exclusively red.

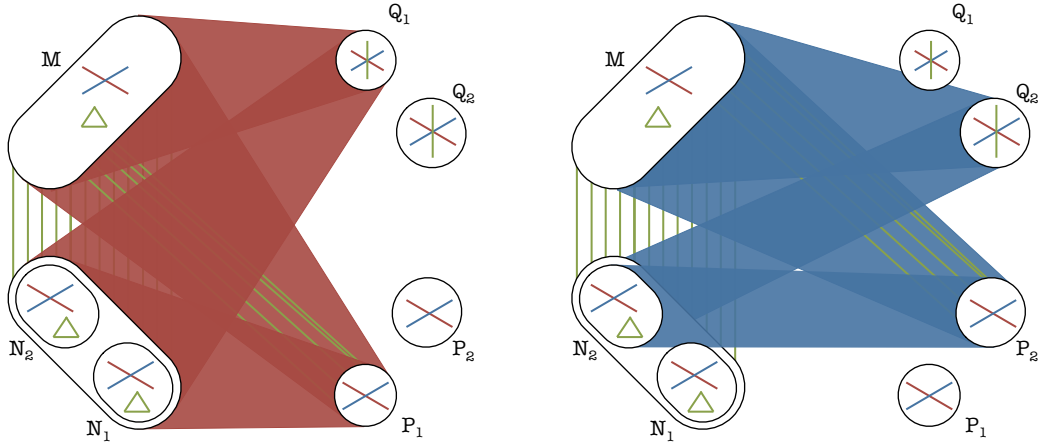


Figure 49: Colouring of $G[M \cup N, P_1 \cup Q_1]$ in Claim 3.6.b.

Recalling (50), we then have

$$\begin{aligned} |N_1| &\geq (\tfrac{1}{2}\alpha_1 - 26\eta^{1/2})k, & |P_1 \cup Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 26\eta^{1/2})k, \\ |N_2| &\geq (\tfrac{1}{2}\alpha_2 - 33\eta^{1/2})k, & |P_2 \cup Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 15\eta^{1/2})k. \end{aligned}$$

Finally, suppose there exists a red matching R_S on at least $54\eta^{1/2}k$ vertices in $G[N_1, P_2 \cup Q_2]$. Then, $|N \setminus V(R_S)|, |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 26\eta^{1/2})k$ so, by Lemma 2.8, there exists a red connected-matching R_L on at least $(\alpha_1 - 54\eta^{1/2})k$ vertices in $G[N \setminus V(R_S), P_1 \cup Q_1]$. The red matchings R_S and R_L share no vertices and, since $G[N]$ is red effectively-connected, belong to the same red component of G , thus, together, they form a red connected-matching on at least $\alpha_1 k$ vertices. Therefore, after discarding at most $27\eta^{1/2}k$ vertices from each of N_1 and $P_2 \cup Q_2$, we may assume that all edges in $G[N, P_2 \cup Q_2]$ are coloured blue, thus completing the proof of Claim 3.6.b. \square

In summary, combining the two cases above, we may now assume that there are no blue edges present in $G[M \cup N, P_1 \cup Q_1]$ and no red edges present in $G[M \cup N, P_2 \cup Q_2]$. In particular, we may assume that all all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and that all edges in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.

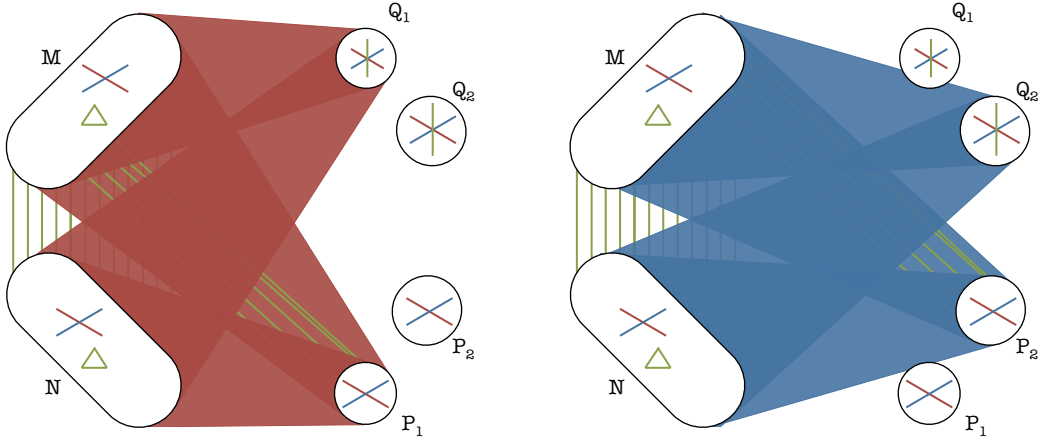


Figure 50: Colouring after Claim 3.6.

We then have

$$\left. \begin{aligned} |N_1| &\geq (\tfrac{1}{2}\alpha_1 - 54\eta^{1/2})k, & |P_1 \cup Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 67\eta^{1/2})k, \\ |N_2| &\geq (\tfrac{1}{2}\alpha_2 - 58\eta^{1/2})k, & |P_2 \cup Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 42\eta^{1/2})k. \end{aligned} \right\} \quad (51)$$

Recalling (E7), having discarded at most $86\eta^{1/2}k$ vertices from N and at most $25\eta^{1/2}k$ vertices from M , we have

$$|M| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 31\eta^{1/2} \right) k, \quad |N| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 93\eta^{1/2} \right) k.$$

Thus, provided that $\eta \leq (\alpha_2/600)^2$, we have

$$|N| \geq |P_1 \cup Q_1| + 400\eta^{1/2}k, \quad |N| \geq |P_2 \cup Q_2| + 400\eta^{1/2}k.$$

Therefore, if there existed either a red matching on $136\eta^{1/2}k$ vertices or a blue matching on $86\eta^{1/2}k$ vertices in $G[N] \cup G[N, M]$, then these could be used together with edges from $G[N, P \cup Q]$ to obtain a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $111\eta^{1/2}k$ vertices from M and $333\eta^{1/2}k$ vertices from N , we may assume that all edges present in $G[N] \cup G[M, N]$ are coloured exclusively green.

We then have

$$\begin{aligned} |M| &\geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 142\eta^{1/2} \right) k, \\ |N| &\geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 426\eta^{1/2} \right) k. \end{aligned}$$

Now, suppose there exists a green matching G_{MP} on $1146\eta^{1/2}k$ vertices in $G[M, P]$. Letting \tilde{N} be any subset of $290\eta^{1/2}k$, by Lemma 2.1, provided $k \geq 1/\eta^2$, there exists a green connected-matching G_N on at least $289\eta^{1/2}k$ vertices in $G[\tilde{N}]$. We then have $|M \setminus V(G_{MP})|, |N \setminus V(G_N)| \geq (\tfrac{1}{2}\alpha_3 - 716\eta^{1/2})k$ and, thus, by Lemma 2.8, we have a green connected-matching G_{MN} on at least $(\alpha_3 - 1434\eta^{1/2})k$ vertices in $G[M, N]$, which shares no vertices with G_{MP} or G_N . Then, since all edges present in $G[M, N]$ are coloured green, together, G_{MP} , G_{MN} and G_N form a green connected-matching on at least $\alpha_3 k$

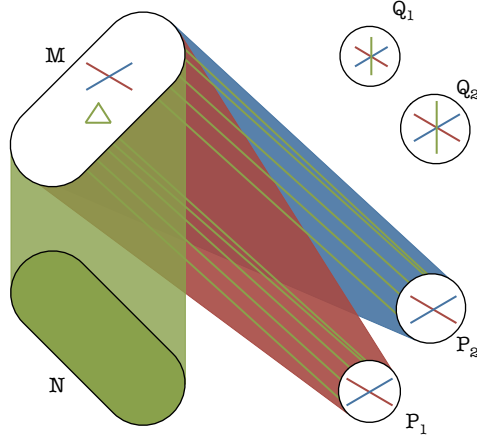


Figure 51: The green graph after considering $G[N] \cup G[M, N]$.

vertices which, since all edges in $G[N]$ are green, is odd. Thus, after discarding at most $574\eta^{1/2}k$ vertices from each of M and P , we may assume that there are no green edges in $G[M, P \cup Q]$. In particular, since earlier we found that there could be no blue edges in $G[M, P_1]$ and no red edges in $G[M, P_2]$, we now know that all edges present in $G[M, P_1]$ are coloured exclusively red and all edges present in $G[M, P_2]$ are coloured exclusively blue.

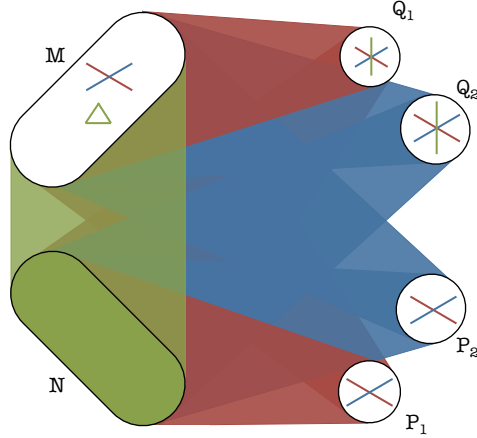


Figure 52: Colouring of $G[M, P]$.

In summary, having discarded these vertices, we may assume that all edges present in $G[M \cup N, P_1 \cup Q_1]$ are coloured exclusively red, that all edges present in $G[M \cup N, P_2 \cup Q_2]$ are coloured exclusively blue, that all edges in $G[N] \cup G[M, N]$ are coloured exclusively green and that we have

$$\begin{aligned} |M| &\geq \left(\max\left\{ \frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3 \right\} - 716\eta^{1/2} \right) k, & |P_1 \cup Q_1| &\geq \left(\frac{1}{2}\alpha_1 - 642\eta^{1/2} \right) k, \\ |N| &\geq \left(\max\left\{ \frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3 \right\} - 426\eta^{1/2} \right) k, & |P_2 \cup Q_2| &\geq \left(\frac{1}{2}\alpha_1 - 616\eta^{1/2} \right) k. \end{aligned}$$

Notice that, provided that $\eta \leq (\alpha_2/5000)^2$, we have $|M| \geq |P_1 \cup Q_1| + 2200\eta^{1/2}k$. Thus, there cannot exist in $G[M]$ a red matching on $1286\eta^{1/2}k$ vertices or a blue matching on $1234\eta^{1/2}k$ vertices. Therefore, after discarding at most $2520\eta^{1/2}k$ vertices from M , we may assume that all edges present

in $G[M]$ are coloured exclusively green and that

$$|M \cup N| \geq (\alpha_3 - 3662\eta^{1/2})k.$$

In summary, we know that all edges present in $G[M \cup N, P_1 \cup Q_1]$ are coloured exclusively red, all edges present $G[M \cup N, P_2 \cup Q_2]$ are coloured exclusively blue and all edges in $G[M \cup N]$ are coloured exclusively green. Thus, we have found, as a subgraph of G , a graph in

$$\mathcal{K} \left(\left(\frac{1}{2}\alpha_1 - 700\eta^{1/2} \right)k, \left(\frac{1}{2}\alpha_1 - 700\eta^{1/2} \right)k, (\alpha_3 - 4000\eta^{1/2})k, 4\eta^4 k \right),$$

thus completing Case E.iii.b.iv.a.

Case E.iii.b.iv.b: $G[M, N \cup P \cup Q_2]$ contains a blue edge.

In this case, following similar steps as in Case E.iii.b.iv.a will result in either a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K} \left(\left(\frac{1}{2}\alpha_1 - 700\eta^{1/2} \right)k, \left(\frac{1}{2}\alpha_1 - 700\eta^{1/2} \right)k, (\alpha_3 - 4000\eta^{1/2})k, 2\eta^4 \right),$$

thus completing Case E.iii.b.iv.a.

Case E.iii.b.iv.c: $G[M, N \cup P \cup Q_1]$ contains no red edges.
and $G[M, N \cup P \cup Q_2]$ contains no blue edges.

Recall that we that we have a decomposition $V(G)$ into $M \cup N \cup P \cup Q$ and a refinement into $M \cup N_1 \cup N_2 \cup P_1 \cup P_2 \cup Q_1 \cup Q_2$, such that

(E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$;

(E6a) $G[M]$, $G[N]$ and $G[P]$ each have a single red and a single blue effective-component;

(E6b) $G[N_1, P_1 \cup Q_1]$ contains a red matching utilising every vertex in $G[N_1 \cup P_1 \cup Q_1]$;

(E6c) all edges present in $G[N_2, P_2 \cup Q_2]$ are coloured exclusively blue.

This decomposition also satisfies

$$\left. \begin{aligned} |M| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 6\eta^{1/2})k, & |N_1| = |P_1 \cup Q_1| &\geq (\frac{1}{2}\alpha_1 - 8\eta^{1/2})k, \\ |N| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 7\eta^{1/2})k, & |N_2|, |P_2 \cup Q_2| &\geq (\frac{1}{2}\alpha_2 - 8\eta^{1/2})k. \end{aligned} \right\} \quad (\text{E7})$$

$$|P| \geq 90\eta^{1/2}k, \quad \max\{|Q_1|, |Q_2|\} \geq 45\eta^{1/2}k. \quad (\text{E8})$$

Additionally, in this case, we may assume that

(E9a) all edges present in $G[Q_2, M]$ are coloured exclusively red;

(E9b) all edges present in $G[Q_1, M]$ are coloured exclusively blue; and

(E9c) all edge present in $G[M, P_1 \cup P_2] \cup G[M, N]$ are green.

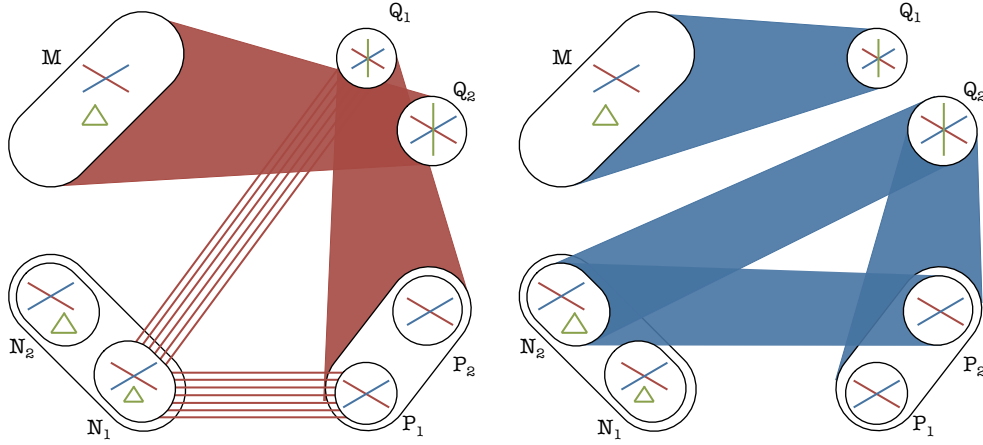


Figure 53: Initial red and blue graphs in Case E.iii.b.iv.c.

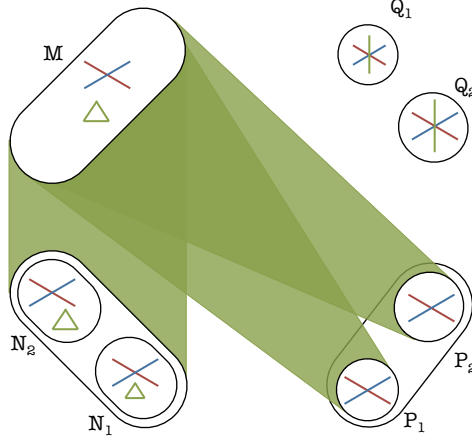


Figure 54: Initial green graph in Case E.iii.b.iv.c.

We begin by proving the following claim which concerns the structure of the red and blue graphs:

Claim 3.7.a. *If $|Q_1| \leq 38\eta^{1/2}k$, we may discard at most $71\eta^{1/2}k$ vertices from N , at most $12\eta^{1/2}k$ vertices from P_1 , at most $59\eta^{1/2}k$ vertices from P_2 and at most $38\eta^{1/2}k$ vertices from Q_1 so that, in what remains, all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.*

Claim 3.6.b. *If $|Q_1| \geq 38\eta^{1/2}k$, we may discard at most $57\eta^{1/2}k$ vertices from N , at most $19\eta^{1/2}k$ vertices from P_1 , at most $38\eta^{1/2}k$ vertices from P_2 and at most $30\eta^{1/2}k$ vertices from Q_1 so that, in what remains, all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.*

Proof. (a) Observe that, since $|Q_1| \leq 38\eta^{1/2}k < 45\eta^{1/2}k$, by (E8), we have $|Q_2| \geq 45\eta^{1/2}k$. Considering the red graph, we show that all edges present in $G[N \cup P, Q_2]$ are blue as follows: Given (E9a), since $|M|, |Q_2| \geq 45\eta^{1/2}k$, by Lemma 2.8, there exists a red connected-matching R_S on at least $88\eta^{1/2}k$ vertices in $G[M, Q_2]$. Then, since $|N| = |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 8\eta^{1/2})k$, recalling (E6a), if $G[N \cup P, Q_2]$ contains a red edge, then $R \cup R_S$ forms a red connected-matching on at least $\alpha_1 k$ vertices. Thus, recalling (E3), all edges present in $G[N \cup P, Q_2]$ are coloured exclusively blue (see Figure 55).

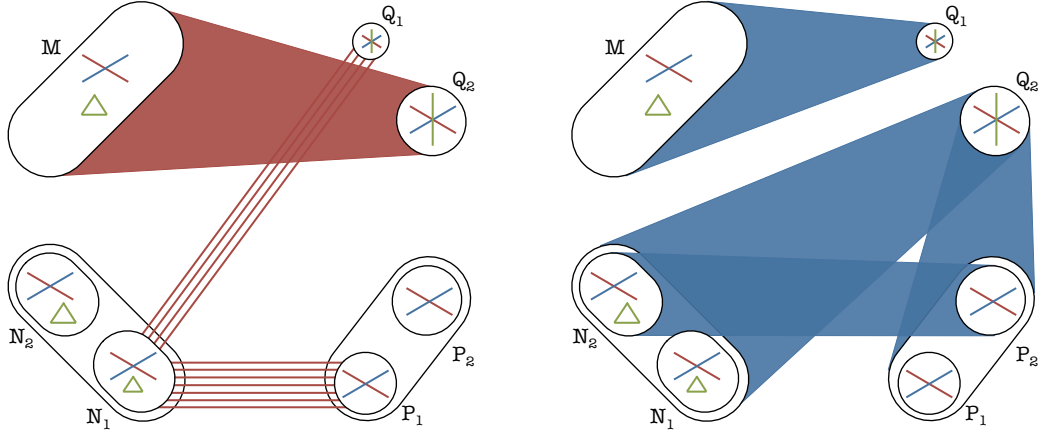


Figure 55: Red and blue subgraphs of G , provided $|Q_1| \leq 38\eta^{1/2}k$.

Now, suppose there exists a blue matching B_T on $24\eta^{1/2}k$ vertices in $G[N, P_1]$ (see Figure 56). Then, since $|Q_2| \geq 45\eta^{1/2}k$, there exist subsets $\tilde{N}_2 \subseteq N_2 \setminus V(B_T)$ and $\tilde{Q}_2 \subseteq Q_2$ such that $|\tilde{N}_2| = |P_2 \cup \tilde{Q}_2| \geq (\frac{1}{2}\alpha_2 - 20\eta^{1/2})k$ and $|Q_2 \setminus \tilde{Q}_2| \geq 11\eta^{1/2}k$. By Lemma 2.8, there exist blue connected-matchings B_U on at least $(\alpha_2 - 42\eta^{1/2})k$ vertices in $G[N_2 \setminus V(B_T), P_2 \cup \tilde{Q}_2]$ and B_V on at least $20\eta^{1/2}k$ vertices in $G[N_1 \setminus V(B_T), Q_2 \setminus \tilde{Q}_2]$. Since all edges present in $G[N, Q_2]$ are coloured blue, B_T , B_U and B_V belong to the same blue component and together, form a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, there can not exist such a matching as B_T and, after discarding at most $12\eta^{1/2}k$ vertices from each of N and P_1 , we may assume that all edges present in $G[N, P_1]$ are coloured exclusively red.

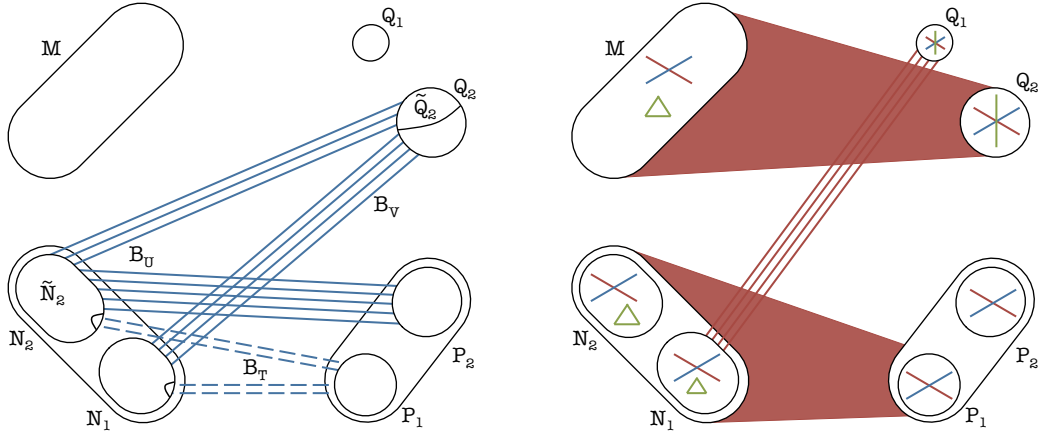


Figure 56: B_T and the resultant colouring of the edges of $G[N, P_2]$.

Having discarded these vertices, we have

$$\left. \begin{aligned} |M| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 6\eta^{1/2})k, & |N_1|, |P_1 \cup Q_1| &\geq (\frac{1}{2}\alpha_1 - 20\eta^{1/2})k, \\ |N| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 19\eta^{1/2})k, & |P_2 \cup Q_2| &\geq (\frac{1}{2}\alpha_2 - 8\eta^{1/2})k, \\ & & |N_2| &\geq (\frac{1}{2}\alpha_2 - 20\eta^{1/2})k. \end{aligned} \right\} \quad (52)$$

Since $|Q_1| \leq 38\eta^{1/2}k$ and $\eta < (\alpha_1/1000)^2$, by (E4b) and (52), we have

$$|N| \geq |P_1| + 200\eta^{1/2}k, \quad |P_1| \geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k.$$

Thus, if there existed a red matching R_S on $118\eta^{1/2}k$ vertices in $G[N, P_2]$, we could obtain a red connected-matching on at least $\alpha_1 k$. Indeed, since $|N \setminus V(R_S)|, |P_1| \geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k$, by Lemma 2.8, there exists a red connected-matching R_L on $118\eta^{1/2}k$ vertices in $G[N \setminus V(R_S), P_1]$. Then, since $G[N]$ has a single red effective-component, R_L and R_S belong to the same red component of G and, therefore, together, form a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $59\eta^{1/2}k$ vertices from each of N and P_2 , we may assume that all edges present in $G[P_2, N]$ are coloured exclusively blue. Finally, we discard all vertices from Q_1 . Having done so, we have $P_1 = P_1 \cup Q_1$ and know that all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red, that all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.

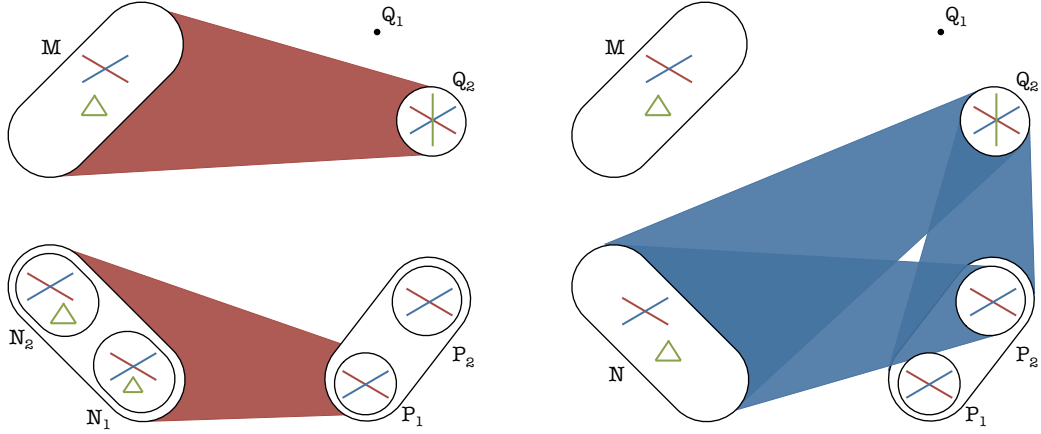


Figure 57: The final red and blue graphs in Claim 3.7.a.

We then have

$$\left. \begin{aligned} |M| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 6\eta^{1/2})k, & |P_1 \cup Q_1| &\geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k, \\ |N| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 78\eta^{1/2})k, & |P_2 \cup Q_2| &\geq (\frac{1}{2}\alpha_2 - 67\eta^{1/2})k. \end{aligned} \right\} \quad (53)$$

thus completing the proof of Claim 3.7.a.

(b) Suppose G is coloured as shown in Figures 53–54 and that we have $|Q_1| \geq 38\eta^{1/2}k$. Considering the blue graph, we are able to show that all edges present in $G[N \cup P, Q_1]$ are red as follows: Given (E9b), since $|M|, |Q_1| \geq 38\eta^{1/2}k$, by Lemma 2.8, there exists a blue connected-matching B_S on at least $74\eta^{1/2}k$ vertices in $G[M, Q_1]$. Then, since $|N_2|, |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 8\eta^{1/2})k$, if $G[N \cup P, Q_1]$ contains a blue edge, we can obtain a blue connected-matching on at least $\alpha_2 k$ vertices utilising edges from B_S and from $G[N_2, P_2 \cup Q_2]$. Thus, recalling (E3), all edges present in $G[N \cup P, Q_1]$ are coloured exclusively red.

Now, let $N_{11} = N_1 \cap V(R)$ and $N_{12} = N_1 \setminus V(R)$. Then, since $|Q_1| \geq 38\eta^{1/2}k$ and $|N_{11}| = |P_1|$, we have $|N_{12}| \geq 38\eta^{1/2}k$. If, there exists a red matching R_S on $18\eta^{1/2}k$ vertices in $G[N_{12}, P_2]$, then, since $|N_2|, |Q_1| \geq (\frac{1}{2}\alpha_1 - 8\eta^{1/2})k - |P_1|$ and all edges in $G[N, Q_1]$ are coloured red, by Lemma 2.8, there exists a red connected-matching R_T on at least $(\alpha_1 - 18\eta^{1/2})k - 2|P_1|$ vertices in $G[N_2, Q_1]$. Since $|N_{11}| = |P_1|$, $R_U = R \cap G[N_{11}, P_1]$ is a red matching on $2|P_1|$ vertices. Thus, since $G[N]$ has a single red effective-component, $R_S \cup R_T \cup R_U$ forms a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $9\eta^{1/2}k$ vertices from each of N_{12} and P_2 , we may assume that all edges in $G[N_{12}, P_2]$ are coloured exclusively blue. We then have $|N_{12}| \geq 28\eta^{1/2}k$ and $|P_2 \cup Q_2| \geq (\alpha_2 k - 17\eta^{1/2})k$.

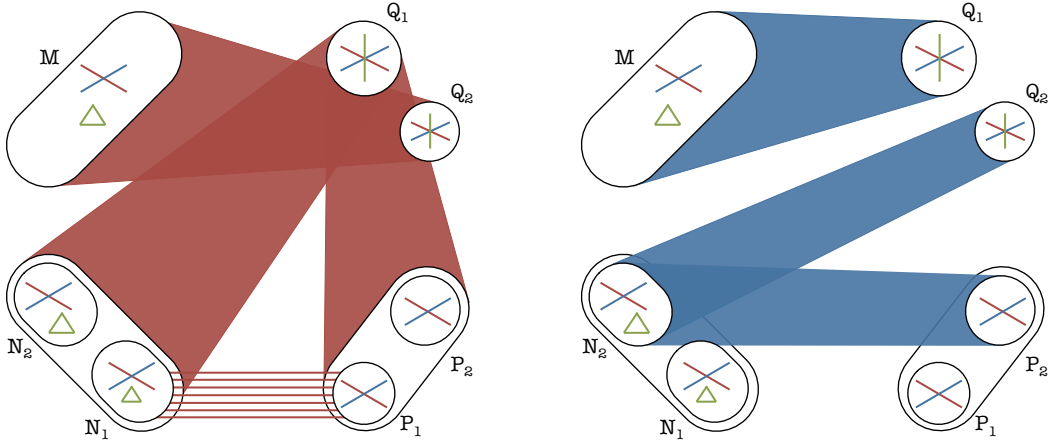


Figure 58: Red and blue subgraphs of G when $|Q_1| \geq 38\eta^{1/2}k$.

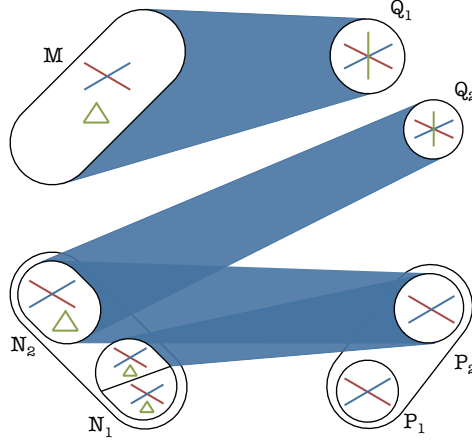


Figure 59: Colouring of $G[N_{12}, P_2]$.

Then, suppose there exists a blue matching B_S on $38\eta^{1/2}k$ vertices in $G[N, P_1]$. Then

$$\begin{aligned} |(N_2 \cup N_{12}) \setminus V(B_S)| &\geq (\tfrac{1}{2}\alpha_2 - 8\eta^{1/2})k + 28\eta^{1/2}k - 19\eta^{1/2}k \\ &\geq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k \geq |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 17\eta^{1/2})k. \end{aligned}$$

Thus, by Lemma 2.8, there exists a blue connected-matching on at least $(\alpha_2 - 38\eta^{1/2})k$ vertices belonging to the same component as B_S . Thus, together, these matchings form a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, after discarding at most $19\eta^{1/2}k$ vertices from each of N and P_1 , we may assume that all edges present in $G[N, P_1]$ are coloured exclusively red and that $|P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 27\eta^{1/2})k$.

Penultimately, suppose there exists a red matching R_V on $58\eta^{1/2}k$ vertices in $G[N_{11}, P_2]$. Then, since $|N \setminus V(R_V)|, |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 27\eta^{1/2})k$, by Lemma 2.8, there exists a red connected-matching R_W on at least $(\alpha_1 - 56\eta^{1/2})k$ vertices in $G[N \setminus V(R_V), P_1 \cup P_2]$, which together with R_V gives a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $29\eta^{1/2}k$ vertices from each of N_{11} and P_2 , we may assume that all edges in $G[N, P_2]$ are coloured exclusively blue.

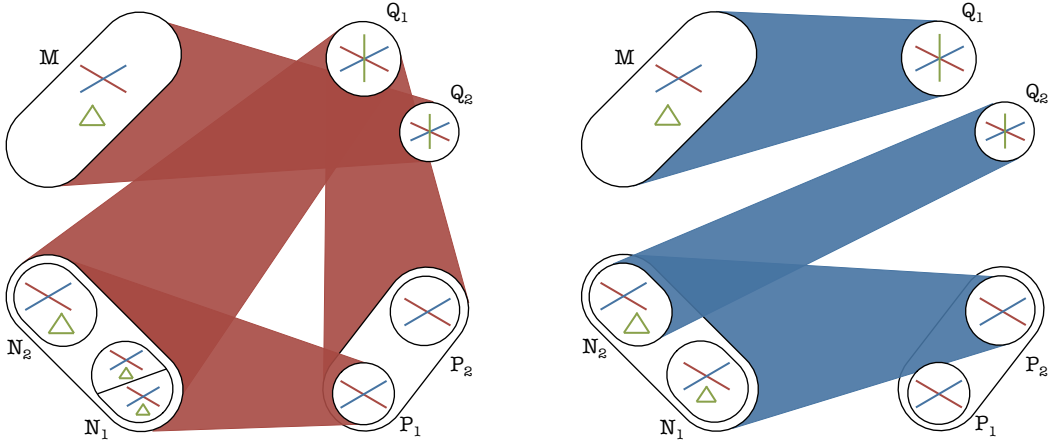


Figure 60: Colouring of $G[N, P]$.

Finally, observe that, if $|Q_2| \geq 30\eta^{1/2}k$, there can be no red edges in $G[N \cup P, Q_2]$. Indeed, in that case, since $|M|, |Q_2| \geq 30\eta^{1/2}k$ and $|N|, |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 27\eta^{1/2})k$, there exist red connected-matchings R_W in $G[M, Q_2]$ and R_X in $G[P_1 \cup Q_1, N]$ which belong to the same component and together span at least $\alpha_1 k$ vertices. Alternatively, if $|Q_2| \leq 30\eta^{1/2}k$, we can discard every vertex in Q_2 , rendering the graph $G[N, Q_2]$ trivial. Thus, in either case, all edges in $G[N, Q_2]$ are coloured exclusively blue.

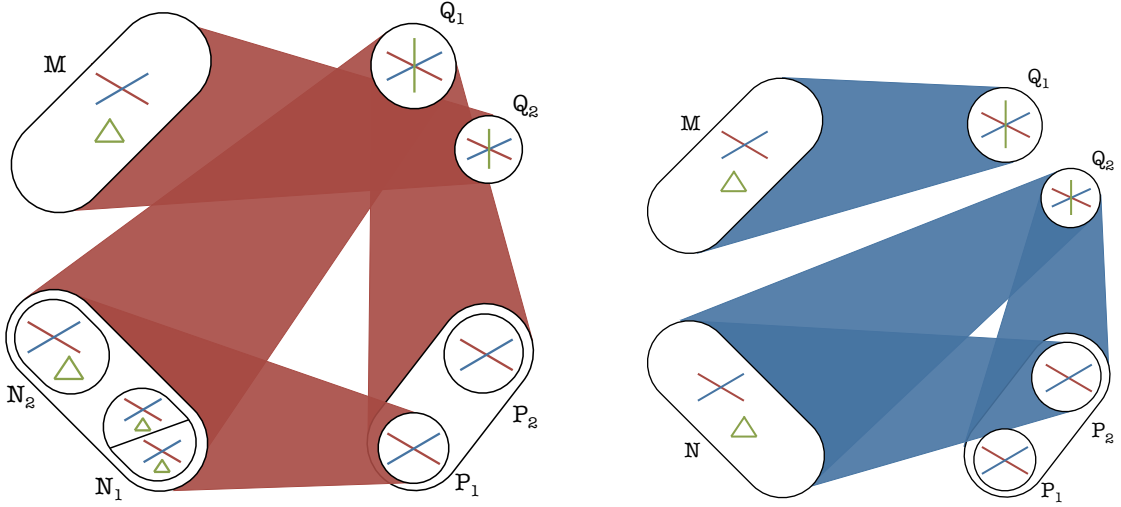


Figure 61: The final red and blue graphs in Claim 3.7.b. if $|Q_2| \geq 30\eta^{1/2}k$.

In summary, having discarded some vertices, we may assume that all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red, that all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue and that

$$\left. \begin{aligned} |M| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 6\eta^{1/2})k, & |P_1 \cup Q_1| &\geq (\frac{1}{2}\alpha_1 - 27\eta^{1/2})k, \\ |N| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 64\eta^{1/2})k, & |P_2 \cup Q_2| &\geq (\frac{1}{2}\alpha_2 - 76\eta^{1/2})k, \end{aligned} \right\} \quad (54)$$

thus, completing the proof of Claim 3.7.b. \square

Having proved the claim, we know that all edges in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and all

edges in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue. Combining (54) and (53), we have

$$\left. \begin{aligned} |M| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 6\eta^{1/2})k, & |P_1 \cup Q_1| &\geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k, \\ |N| &\geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 78\eta^{1/2})k, & |P_2 \cup Q_2| &\geq (\frac{1}{2}\alpha_2 - 76\eta^{1/2})k. \end{aligned} \right\} \quad (55)$$

We now consider the green graph (see Figure 54). Recall that all edges present in $G[M, N \cup P]$ are coloured green. Now, suppose there exists a green matching G_S on $17\eta^{1/2}k$ vertices in $G[N]$. Since $|M| \geq |P| \geq 90\eta^{1/2}k$, by Lemma 2.8, there exists a green connected-matching G_T on $178\eta^{1/2}k$ vertices in $G[M, P]$. Then, by (55), we have

$$|M \setminus V(G_T)|, |N \setminus V(G_S)| \geq (\frac{1}{2}\alpha_3 - 96\eta^{1/2})k.$$

Thus, by Lemma 2.8, there exists a green connected-matching G_L on at least $(\alpha_3 - 194\eta^{1/2})k$ in $G[M \setminus V(G_T), N \setminus V(G_S)]$. Then, since all edges present in $G[M, N]$ are coloured exclusively green, G_S, G_T and G_L belong to the same green component of G and, thus, together form an green odd connected-matching on at least $\alpha_3 k$ vertices. Therefore, after discarding at most $17\eta^{1/2}k$ vertices from N , we may assume that there are no green edges in $G[N]$ and that

$$|N| \geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 95\eta^{1/2})k. \quad (56)$$

We continue to consider $G[N]$. Recall that we know that all edges in $G[N, P_1 \cup Q_1]$ are coloured exclusively red, that all edges in $G[N, P_2 \cup Q_2]$ are coloured exclusively red and that. From (55),

$$|P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k, \quad |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 76\eta^{1/2})k.$$

Recalling (E4b) and (56), provided that $\eta \leq (\alpha_1/2000)^2$, we have

$$|N| \geq |P_1 \cup Q_1| + 300\eta^{1/2}k.$$

Then, suppose that there exists a red-matching R_A on at least $118\eta^{1/2}k$ vertices in $G[N]$. Then, since $|N \setminus V(R_A)| \geq |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k$, by Lemma 2.8, there exists a red connected-matching R_B on at least $(\alpha_1 - 118\eta^{1/2})k$ vertices in $G[N \setminus V(R_A), P_1 \cup Q_1]$. Then, by (E6a), R_A and R_B belong to the same red component of G and thus, together, form a red connected-matching on at least $\alpha_1 k$ vertices. Likewise, if there exists a blue-matching B_A on at least $154\eta^{1/2}k$ vertices in $G[N]$, then we can obtain a blue connected-matching on at least $\alpha_2 k$ vertices, as follows: Since $|N \setminus V(R_B)| \geq |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_1 - 76\eta^{1/2})k$, by Lemma 2.8, there exists a blue connected-matching B_B on at least $(\alpha_1 - 154\eta^{1/2})k$ vertices in $G[N \setminus V(B_A), P_2 \cup Q_2]$. Then, since all edges present in $G[N, Q_2]$ are coloured exclusively blue, B_A and B_B , together, form a blue connected-matching on at least $\alpha_2 k$ vertices.

Thus, after discarding at most a further $272\eta^{1/2}k$ vertices from N , we can assume that there are no edges of any colour in $G[N]$ and that

$$|N| \geq (\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 367\eta^{1/2})k \geq 10\eta^{1/2}k.$$

This contradicts the fact that G is $4\eta^4 k$ -almost-complete, thus completing Case E.iii.b.iv.c. and the proof of Theorem B. \square

References

- [Dir52] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc.* (3), 2:69–81, 1952.
- [EG59] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.* 10:337–356 (unbound insert), 1959.
- [Fer15a] D. G. Ferguson. The Ramsey number of mixed-parity cycles I. Preprint at [arXiv \[math.CO\]](#), 2015.
- [Fer15c] D. G. Ferguson. The Ramsey number of mixed-parity cycles III. Preprint at [arXiv \[math.CO\]](#), 2015.
- [FL07a] A. Figaj and T. Łuczak. The Ramsey number for a triple of long even cycles. *J. Combin. Theory Ser. B*, 97(4):584–596, 2007.
- [FL07b] A. Figaj and T. Łuczak. The Ramsey number for a triple of large cycles. Preprint at [arXiv:0709.0048 \[math.CO\]](#), 2007.
- [KSS09] Y. Kohayakawa, M. Simonovits, and J. Skokan. Stability of the Ramsey number of cycles. Manuscript, 2009.